LVIII. *The Combustion Problem of Internal Ballistics.—*
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### IV. SOLUTION BY SELECTOR VARIABLE SEPARATION.

14. *Selector Variable Method.*—The parametric solution
developed in Section III. exhibits conditions in a way such
that propellant effects may be examined in detail, whilst the
method of attack now to be discussed enables the effects of
gun influences to be ascertained. Where for the previous
solution the functions V, X, P are set out as series in
ascending powers of the $\epsilon$ parameters, here they will be
developed in terms of selector variables, these being
$(1, 0) = z$, the fraction of propellant size consumed at any
instant, $(1, 1)$, the variable proportional to the fraction of
charge burnt, $\left(0, \dfrac{M}{V}\right)$, corresponding effectively to the
reciprocal of the shot travel for the condition $(1, 0)$.
The forms of the equations lead to expansions for V, X, P
as double series, the leading terms of which involve powers
of the selector variables, the subsidiary series attaching to
the leading terms involving powers of $(1, -1)$ only, the
subsidiary variable arising from the form in which $dV/dz$ is
shown explicitly as a function of V, P, $(-1, -1)$.

The equations to be solved are

$$\frac{dV}{dz} = 1 - \epsilon_1(-1, -1)V^2 + \epsilon_2 P, \quad . \quad . \quad (18)$$

$$\frac{dX}{dz} = MV\{(-1, -1)X + \epsilon_2 - \epsilon_3\}, \quad . \quad . \quad (19)$$

$$P = \frac{(1, 1) - \epsilon_1 V^2}{X - \epsilon_3(1, 1)}. \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Writing

$$Z = \left(0, \frac{M}{\nu}\right), \quad . \quad . \quad . \quad . \quad . \quad (47)$$

and for the moment neglecting expressions containing the $\epsilon$ terms, provisional solutions to the equations are

$$V = (1, 0),$$

$$ZX = 1,$$

$$P = (1, 1)Z.$$

Thus all the leading terms will be functions of $(1, 0)$, $(1, 1)$, and Z. From the form $\{1 - \epsilon_1(-1, -1)V^2\}$, in which the first term of V is $(1, 0)$, the variable $(1, -1)$ comes into play and gives rise to the subsidiary series in powers of this variable attaching to the leading terms of the main series. We may therefore assume as provisional solutions, from the non-linearity of the general equation, series in powers of Z in the following forms :—

$$V = V_0 + V_1 Z + V_2 Z^2 + V_3 Z^3 + V_4 Z^4 + \ldots, \quad (48)$$

$$\log ZX = X_0 + X_1 Z + X_2 Z^2 + X_3 Z^3 + X_4 Z^4 + \ldots, \quad (49)$$

$$P = P_1 Z + P_2 Z^2 + P_3 Z^3 + P_4 Z^4 + \ldots, \quad . \quad . \quad (50)$$

where the coefficients of the powers of Z are functions of $z = (1, 0)$, whose explicit forms remain to be determined [*]. It will be seen that equation (49) is in effect an asymptotic expansion for X in powers of Z, since $Z = \left(0, \dfrac{M}{\nu}\right)$ becomes small for large M.

From equation (48)

$$V^2 = V_0{}^2 + 2V_0 V_1 Z + (2V_0 V_2 + V_1{}^2)Z^2 + (2V_0 V_3 + 2V_1 V_2)Z^3$$
$$+ (2V_0 V_4 + V_1 V_3 + V_2{}^2)Z^4 + \ldots, \quad . \quad (51)$$

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[*] Again, no confusion need arise between the present notation and that of preceding sections.

and

$$\frac{dV}{dz} = \frac{dV_0}{dz} + \left\{ \frac{dV_1}{dz} - M(0, -1)V_1 \right\} Z$$
$$+ \left\{ \frac{dV_2}{dz} - 2M(0, -1)V_2 \right\} Z^2$$
$$+ \left\{ \frac{dV_3}{dz} - 3M(0, -1)V_3 \right\} Z^3 + \dots ; \qquad (52)$$

∴ from equations (18), (50), (51), (52), equating coefficients of powers of Z,

$$\frac{dV_0}{dz} = 1 - \epsilon_1(-1, -1)V_0^2, \quad \dots \quad \dots \quad (53)(\text{i.})$$

$$\frac{dV_1}{dz} = M(0, -1)V_1 - \epsilon_1(-1, -1)2V_0V_1 + \epsilon_2 P_1, \qquad (53)(\text{ii.})$$

$$\frac{dV_2}{dz} = 2M(0, -1)V_2 - \epsilon_1(-1, -1)(2V_0V_2 + V_1^2) + \epsilon_2 P_2,$$
$$\dots \quad (53)(\text{iii.})$$

$$\frac{dV_3}{dz} = 3M(0, -1)V_3 - \epsilon_1(-1, -1)(2V_0V_3 + 2V_1V_2) + \epsilon_2 P_3,$$
$$\dots \quad (53)(\text{iv.})$$

$$\frac{dV_4}{dz} = 4M(0, -1)V_4 - \epsilon_1(-1, -1)(2V_0V_4 + 2V_1V_3 + V_2^2)$$
$$+ \epsilon_2 P_4. \quad (53)(\text{v.})$$

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Equation (19) may be written

$$\frac{d}{dz}\{\log ZX\} = -M(0, -1) + M(-1, -1)V + \frac{M(\epsilon_2 - \epsilon_3)VZ}{ZX},$$
$$\dots \quad (54)$$

whence, following a similar procedure,

$$\frac{dX_0}{dz} = -M(0, -1) + M(-1, -1)V_0, \quad \dots \quad \dots \quad (55)(\text{i.})$$

$$\frac{dX_1}{dz} = M(0, -1)X_1 + M(-1, -1)V_1 + M(\epsilon_2 - \epsilon_3)e^{-X_0}V_0,$$
$$\dots \quad (55)(\text{ii.})$$

$$\frac{dX_2}{dz} = 2M(0, -1)X_2 + M(-1, -1)V_2$$
$$+ M(\epsilon_2 - \epsilon_3)e^{-X_0}(V_1 - V_0X_1), \quad (55)(\text{iii.})$$

$$\frac{dX_3}{dz} = 3M(0, -1)X_3 + M(-1, -1)V_3 + M(\epsilon_2 - \epsilon_3)e^{-X_0}$$
$$\left\{ V_2 - V_1X_1 + V_0\left(\frac{X_1^2}{2} - X_2\right) \right\}, \quad (55)(\text{iv.})$$

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$$\frac{dX_4}{dz} = 4M(0, -1)X_4 + M(-1, -1)V_4$$
$$+ M(\epsilon_2 - \epsilon_3)e^{-X_0}\left\{V_3 - V_2X_1 + V_1\left(\frac{X_1^2}{2} - X_2\right)\right.$$
$$\left. - V_0\left(\frac{X_1^3}{6} - X_1X_2 + X_3\right)\right\}. \quad . \quad . \quad (55)(v.)$$

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Finally, from equation (20)

$$P_1 = \{(1, 1) - \epsilon_1 V_0^2\}e^{-X_0}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)(i.)$$

$$P_2 = -\epsilon_1 2V_0V_1 e^{-X_0} - \{X_1 - \epsilon_3(1, 1)e^{-X_0}\}P_1, \quad . \quad . \quad (56)(ii.)$$

$$P_3 = -\epsilon_1(2V_0V_2 + V_1^2)e^{-X_0} - \left(\frac{X_1^2}{2} + X_2\right)P_1$$
$$- \{X_1 - \epsilon_3(1, 1)e^{-X_0}\}P_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad (56)(iii.)$$

$$P_4 = -\epsilon_1(2V_0V_3 + 2V_1V_2)e^{-X_0} - \left(\frac{X_1^3}{6} + X_1X_2 + X_3\right)P_1$$
$$- \left(\frac{X_1^2}{2} + X_2\right)P_2 - \{X_1 - \epsilon_3(1, 1)e^{-X_0}\}P_3. \quad . \quad (56)(iv.)$$

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15. *Functions of Series.*—Before proceeding to the solution of equations (53), (55), (56), a general method will be established for expressing products of functions of convergent series in ascending powers of a variable in terms of one series in ascending powers of the variable.

Generalizing the Jolliffe procedure, let

$$A = a_0 + \frac{a_1}{1!}w + \frac{a_2}{2!}w^2 + \ldots + \frac{a_r}{r!}w^r + \ldots,$$

$$B = b_0 + \frac{b_1}{1!}w + \frac{b_2}{2!}w^2 + \ldots + \frac{b_r}{r!}w^r + \ldots,$$

$$C = c_0 + \frac{c_1}{1!}w + \frac{c_2}{2!}w^2 + \ldots + \frac{c_r}{r!}w^r + \ldots,$$

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the series A, B, C... being convergent. Let $\phi(A)$ be any function of A, $\chi(B)$ any function of B, $\psi(C)$ any function of C, and so on. Also let

$$\phi(A)\chi(B)\psi(C)\ldots = S,$$

where

$$S = S_0 + \frac{S_1}{1!}w + \frac{S_2}{2!}w^2 + \ldots + \frac{S_r}{r!}w^r + \ldots.$$

Consider the operator

$$\Omega \equiv \left\{ a_1 \frac{\partial}{\partial a_0} + a_2 \frac{\partial}{\partial a_1} + a_3 \frac{\partial}{\partial a_2} + \dots + b_1 \frac{\partial}{\partial b_0} + b_2 \frac{\partial}{\partial b_1} \right.$$
$$\left. + b_3 \frac{\partial}{\partial b_2} + \dots + c_1 \frac{\partial}{\partial c_0} + c_2 \frac{\partial}{\partial c_1} + c_3 \frac{\partial}{\partial c_2} + \dots \right\},$$

where the $a$'s, $b$'s, and $c$'s are assumed to commute. Then

$$\Omega S = \chi(B)\psi(C)\dots \frac{\partial \phi(A)}{\partial A} \left\{ a_1 + \frac{a_2}{1!}w + \frac{a_3}{2!}w^2 + \dots \right\}$$
$$+ \phi(A)\psi(C)\dots \frac{\partial \chi(B)}{\partial B} \left\{ b_1 + \frac{b_2}{1!}w + \frac{b_3}{2!}w^2 + \dots \right\}$$
$$+ \dots \dots \dots \dots \dots \dots \dots$$
$$= \chi(B)\psi(C)\dots \frac{d}{dw}\phi(A) + \phi(A)\psi(C)\dots \frac{d}{dw}\chi(B) + \dots$$
$$= \frac{dS}{dw}$$
$$= S_1 + \frac{S_2}{1!}w + \frac{S_3}{2!}w^2 + \dots + \frac{S_r}{(r-1)!}w^{r-1} + \dots,$$

leading to

$$S_0 = \phi(a_0)\chi(b_0)\psi(c_0)\dots,$$
$$S_1 = \Omega S_0,$$
$$S_2 = \Omega S_1,$$
$$\dots \dots \dots$$
$$S_r = \Omega S_{r-1},$$

giving S as an expansion in powers of $w$.

The procedure is generally applicable to equations (53), (55), (56), but is most useful in evaluating the P functions, where, for example,

$$\mathbb{P}_4 = \begin{vmatrix} -\epsilon_1(2V_0V_3 + 2V_1V_2)e^{-X_0}, & X_1 - \epsilon_3(1,1)e^{-X_0}, & \frac{X_1^2}{2} + X_2, & \frac{X_1^3}{6} + X_1X_2 + X_3, \\ -\epsilon_1(2V_0V_2 + V_1^2)e^{-X_0}, & 1, & X_1 - \epsilon_3(1,1)e^{-X_0}, & \frac{X_1^2}{2} + X_2, \\ -\epsilon_1 2V_0V_1 e^{-X_0}, & 0, & 1, & X_1 - \epsilon_3(1,1)e^{-X_0}, \\ \{(1,1) - \epsilon_1 V_0^2\}e^{-X_0}, & 0, & 0, & 1, \end{vmatrix}$$

where the V and X functions are known series in ascending powers of the independent variable.

16. *Convergency Considerations.*—The explicit formulation of the functions satisfying the sets of subsidiary equations

(53), (55), (56) depends upon the form taken by $V_0$, as the solution of equation (53) (i.) is the starting-point for subsequent developments. All the coefficients of the powers of the selector variable in the forms of solution indicated in equations (48), (49), (50) depend basically upon $V_0$ and sums of powers and products of $V_0$ subsequently integrated. Now Z is a function of $z$, taking the value 1 for $z=0$, and having values decreasing steadily with $z$ increasing to 1. The series, regarded as functions of Z, are in consequence convergent, provided that it can be shown that the co-efficients of powers of Z are themselves convergent series whose respective sums gradually diminish in numerical magnitude. We have in effect a solution in which the coefficients of the various powers of the variables can be exhibited in array form. It is of importance therefore to investigate in detail the precise nature of the coefficient $V_0$. $V_0 = 0$ for $z = 0$, and the first term in $V_0$ is $z$. This will be the leading term, to which will be attached a series in powers of some variable or parameter to be ascertained, the form of which will govern the nature of the subsidiary series attaching to leading terms weighted by powers of the selector variable Z.

Write $V_0 = z\psi$, where $\psi$ is the function whose intrinsic nature needs to be investigated. Then, writing $u = (1, -1)$, from equation (53) (i.)

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi}{1 + \nu u} \right\} = 1 - \epsilon_1 u \psi^2, \quad . \quad . \quad (57)$$

and, consequently, if $\psi$ can be expanded as a series it may be expressed in ascending powers of $u$ or $\nu$ or $\epsilon_1$, and the other series weighting subsequent leading coefficients may be similarly treated.

Consider first $\psi$ as a series in powers of $\epsilon_1$, and write

$$\psi = \psi_0 - \epsilon_1 \psi_1 + \epsilon_1^2 \psi_2 - \epsilon_1^3 \psi_3 + \dots,$$

where the $\psi$'s will be functions of $\nu$ and $u$.

Then

$$\psi^2 = \psi_0^2 - \epsilon_1 2\psi_0\psi_1 + \epsilon_1^2 (2\psi_0\psi_2 + \psi_1^2) \\ - \epsilon_1^3 (2\psi_0\psi_3 + 2\psi_1\psi_2) + \dots,$$

and from equation (57), equating coefficients of powers of $\epsilon_1$,

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_0}{1 + \nu u} \right\} = 1, \quad . \quad . \quad . \quad . \quad (58)(\text{i.})$$

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_1}{1 + \nu u} \right\} = u\psi_0^2, \quad . \quad . \quad . \quad (58)(\text{ii.})$$

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_2}{1 + \nu u} \right\} = u 2 \psi_0 \psi_1, \qquad . \quad . \quad (58)(\text{iii.})$$

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_3}{1 + \nu u} \right\} = u(2\psi_0 \psi_2 + \psi_1{}^2). \quad (58)(\text{iv.})$$

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From equation (58) (i.) $\psi_0 = 1$, to conform to starting conditions. From equation (58) (ii.) $\psi_1$ is positive, since it is formed from the summation of positive quantities through a positive range. From equation (58) (iii.) $\psi_2$ is positive, and so on for all the $\psi$ functions. Further, from inspection, $\psi_1$ starts with a term in $u$ having a positive coefficient, $\psi_2$ with a term in $u^2$ with a positive coefficient, and generally $\psi_r$ with a term in $u^r$ with an attached positive coefficient.

From equation (58) (ii.), for $\nu u < 1$,

$$\frac{d}{du} \left\{ \frac{u\psi_1}{1 + \nu u} \right\} = \frac{u}{(1 + \nu u)^2} = u\{1 - 2\nu u + 3\nu^2 u^2 - 4\nu^3 u^3 + \ldots\},$$

whence, integrating,

$$\psi_1 = u(1 + \nu u) \left\{ \frac{1}{2} - \frac{2}{3}\nu u + \frac{3}{4}\nu^2 u^2 - \frac{4}{5}\nu^3 u^3 + \ldots \right\}$$

$$= u \left\{ \frac{1}{2} - \frac{1}{2 \cdot 3}\nu u + \frac{1}{3 \cdot 4}\nu^2 u^2 - \frac{1}{4 \cdot 5}\nu^3 u^3 + \ldots \right\}.$$

Thus $\psi_1$ is absolutely and uniformly convergent, and by a method similar to that applied in paragraph (11) it can be shown that $\psi_1 < \frac{1}{2}u$ for $\nu u = 1$.

From equation (58)(iii.)

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_2}{1 + \nu u} \right\} = u 2 \psi_0 \psi_1 < u^2,$$

and following an argument similar to the foregoing $\psi_2 < \frac{1}{3}u^2$.

From equation (58)(iv.)

$$(1 + \nu u)^2 \frac{d}{du} \left\{ \frac{u\psi_3}{1 + \nu u} \right\} = u(2\psi_0\psi_2 + \psi_1{}^2) < \frac{11}{12}u^3,$$

and it follows that $\psi_3 < \dfrac{11}{48}u^3$. Similarly it can be shown that

$$\psi_4 < \frac{19}{120}u^4 \text{ and } \psi_5 < \frac{473}{4320}u^5.$$

Assuming up to $\psi_{r-1}$ that $\psi_s < u^s$, where $(r-1) \nless s \nless 0$, then since

$$(1+vu)^2 \frac{d}{du} \left\{ \frac{u\psi_r}{1+vu} \right\} = u(\psi_0\psi_{r-1} + \psi_1\psi_{r-2} + \ldots$$
$$+ \psi_{r-2}\psi_1 + \psi_{r-1}\psi_0),$$

and there are altogether $r$ products within the bracket on the right-hand side, each being positive and less than $u^{r-1}$, it follows that

$$\psi_r < \frac{r}{r+1} u^r < u^r.$$

Hence $\psi_r < u^r$ in all cases, and in consequence $\psi$, when expanded in powers of $\epsilon_1$, contains terms which are less in absolute value than corresponding terms in the expansion

$$\bar{\psi} = 1 - \epsilon_1 u + \epsilon_1^2 u^2 - \epsilon_1^3 u^3 + \ldots .$$

The upper limit to the value of $u$ is 2, and $\epsilon_1 = RM\Sigma/2s$, taking values in practice never reaching 1/2. $\bar{\psi}$ is therefore absolutely and uniformly convergent, and *à fortiori*, $\psi$. It will be seen that $\bar{\psi}$ approximates to $\dfrac{1}{(1+\epsilon_1 u)}$, and is in consequence always $< 1$.

$V_0$ is the main term in the expansion for V, to which are added minor quantities from the subsidiary terms. Hence for $z=1$, *i. e.*, at the end of propellant combustion, under certain conditions, such as firing with large sizes of propellant or with reduced charge weights, the value of V tends to be less than unity, illustrating the physical effect of appreciable propellant gas cooling. In general, however, the value of V approximates closely to $z$, through the compensating additive effect of the terms subsidiary to $V_0$.

The convergency argument followed above applies to all the subsequent expanded series attaching to the leading terms of the main V, X, P series, since the mode of formation observes a similar procedure arising out of the subsidiary sets of differential equations required to investigate the explicit forms, repeated integrations conserving the absolute convergency established for the series weighting the first leading term in V. The various series sum to quantities of approximately the same orders of magnitude, and as these load the leading terms of the expansions, which themselves diminish rapidly in absolute magnitude, the general series derived for V, X, P are absolutely and uniformly convergent, the convergency in practice being rapid.

17. *Explicit Forms of the Subsidiary Series and of the Leading Terms of the Main Series* —Since $\psi$ is an absolutely

and uniformly convergent series the terms may be re-arranged in any order without affecting the value, and for purposes of subsequent development and application it is essential to consider the various forms in which $\psi$ may be expanded. For convergency considerations $\psi$ has been expressed as a series in ascending powers of $\epsilon_1$. Equation (57) indicates that it may also be expanded in ascending powers of $\nu$ or in ascending powers of $u$, the latter form, since $u = (1, -1)$, implying further the possible expression as a series in ascending powers of $z$, $\mid \nu z \mid$ never being $> \frac{1}{2}$. All four modes have their uses. The expansion in powers of $\epsilon_1$ may be directly derived by a treatment following that indicated under Section III., ending at powers of $\epsilon_1$, indicating that the retention of terms of higher order is unnecessary. The expansion in powers of $\nu$ discloses effects pertaining to the influence of the assumed shape of propellant grain, since, for example, for $\nu = 0$ the terms outstanding contain only functions of $\epsilon_1$ and $z$, whilst for shapes such as flat strip, where $\nu$ is small, only the terms containing low powers of $\nu$ need be retained. The expansion in powers of $z$, while reasonably convergent, leads to cumbrous coefficients; in consequence the treatment will be restricted to the expansion of the subsidiary series in ascending powers of $(1, -1)$.

All the subsidiary series of the $\psi$ type for V, X, P will follow the form established for the subsidiary series attaching to the leading term in $V_0$. Hence to ascertain the algebraic forms of the leading series it will be necessary only to consider the solution of the subsidiary equations in so far as concerns the indices attaching to the various powers of $(1, 0)$ and of $(0, 1)$, the precise forms of the coefficients being ignored temporarily. With this treatment terms in powers of $(1, 0)$ higher than the lowest retained will look after themselves, their precise nature being ascertained when the full analysis is applied.

Consideration of equations (53), (55), (56) leads to the following grouping of the leading suffixed terms for V, X, P :—

|   | V. | X. | P. |
|---|---|---|---|
| 0 ......... | $(1, 0)$ | $(2, -2)$ | — |
| 1 ......... | $(2, 1)$ | $(2, 0)$ | $(1, 1)$ |
| 2 ......... | $(3, 2)$ | $(3, 1)$ | $(2, 2)$ |
| 3 ......... | $(4, 3)$ | $(4, 2)$ | $(3, 3)$ |
| 4 ......... | $(5, 4)$ | $(5, 3)$ | $(4, 4)$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $r$ ......... | $(r+1, r)$ | $(r+1, r-1)$ | $(r, r)$ |

The explicit forms for V, log ZX, P are therefore

$$V = (1, 0)\{(v00) + (v01)(1, -1)$$
$$+ (v02)(2, -2) + (v03)(3, -3) + \ldots\}$$
$$+ \left(2, \frac{M}{\nu} + 1\right)\{(v10) + (v11)(1, -1)$$
$$+ (v12)(2, -2) + (v13)(3, -3) + \ldots\}$$
$$+ \left(3, \frac{2M}{\nu} + 2\right)\{(v20) + (v21)(1, -1)$$
$$+ (v22)(2, -2) + (v23)(3, -3) + \ldots\}$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (59)$$

$$\log ZX = (2, -2)\{x00) + (x01)(1, -1)$$
$$+ (x02)(2, -2) + (x03)(3, -3) + \ldots\}$$
$$+ \left(2, \frac{M}{\nu}\right)\{(x10) + (x11)(1, -1)$$
$$+ (x12)(2, -2) + (x13)(3, -3) + \ldots\}$$
$$+ \left(3, \frac{2M}{\nu} + 1\right)\{(x20) + (x21)(1, -1)$$
$$+ (x22)(2, -2) + (x23)(3, -3) + \ldots\}$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

$$P = \left(1, \frac{M}{\nu} + 1\right)\{(p10) + (p11)(1, -1)$$
$$+ (p12)(2, -2) + (p13)(3, -3) + \ldots\}$$
$$+ \left(2, \frac{2M}{\nu} + 2\right)\{(p20) + (p21)(1, -1)$$
$$+ (p22)(2, -2) + (p23)(3, -3) + \ldots\}$$
$$+ \left(3, \frac{3M}{\nu} + 3\right)\{(p30) + (p31)(1, -1)$$
$$+ (p32)(2, -2) + (p33)(3, -3) + \ldots\}$$
$$+ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad , \quad (61)$$

where $(vrs)$, $(xrs)$, and $(prs)$ are the coefficients to be determined.

18. *Derivation of the Coefficients.*—Substitution of the preceding forms in the appropriate subsidiary equations leads to the following explicit values :

$$V_0 :$$
$$(v00) = 1,$$

$$(v01) = -\frac{1}{2}\,\epsilon_1,$$

$$(v02) = \frac{1}{3}\,\epsilon_1{}^2 + \nu\,\frac{1}{6}\,\epsilon_1,$$

$$(v03) = -\frac{11}{48}\,\epsilon_1{}^3 - \nu\,\frac{1}{4}\,\epsilon_1{}^2 - \nu^2\,\frac{1}{12}\,\epsilon_1,$$

$$(v04) = \frac{19}{120}\,\epsilon_1{}^4 + \nu\,\frac{13}{48}\,\epsilon_1{}^3 + \nu^2\,\frac{11}{60}\,\epsilon_1{}^2 + \nu^3\,\frac{1}{20}\,\epsilon_1.$$

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$V_1$ :

$$(v10) = \frac{1}{2}\,\epsilon_2,$$

$$(v11) = -\frac{2}{3}\,\epsilon_1\epsilon_2 + \nu\,\frac{1}{6}\,\epsilon_2 + M\,\frac{1}{6}\,\epsilon_2,$$

$$(v12) = \frac{17}{24}\,\epsilon_1{}^2\epsilon_2 - \nu\,\frac{1}{12}\,\epsilon_1\epsilon_2 - M\left(\frac{3}{16}\,\epsilon_1\epsilon_2 - \nu\,\frac{1}{24}\,\epsilon_2\right) + M^2\,\frac{1}{24}\,\epsilon_2,$$

$$(v13) = -\frac{2}{3}\,\epsilon_1{}^3\epsilon_2 - \nu\,\frac{7}{40}\,\epsilon_1{}^2\epsilon_2 + \nu^2\,\frac{1}{60}\,\epsilon_1\epsilon_2$$
$$+ M\left(\frac{8}{45}\,\epsilon_1{}^2\epsilon_2 - \nu\,\frac{29}{720}\,\epsilon_1\epsilon_2 - \nu^2\,\frac{1}{120}\,\epsilon_2\right)$$
$$- M^2\,\frac{13}{240}\,\epsilon_1\epsilon_2 + M^3\,\frac{1}{120}\,\epsilon_2.$$

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$V_2$ :

$$(v20) = \frac{1}{3}\,\epsilon_2\epsilon_3,$$

$$(v21) = -\frac{5}{16}\,\epsilon_1\epsilon_2{}^2\,\frac{5}{12}\,\epsilon_1\epsilon_2\epsilon_3 + \nu\,\frac{1}{6}\,\epsilon_2\epsilon_3 - M\left(\frac{3}{16}\,\epsilon_2{}^2 - \frac{7}{24}\,\epsilon_2\epsilon_3\right),$$

$$(v22) = \frac{5}{8}\,\epsilon_1{}^2\epsilon_2{}^2 + \frac{13}{30}\,\epsilon_1{}^2\epsilon_2\epsilon_3 - \nu\left(\frac{13}{80}\,\epsilon_1\epsilon_2{}^2 + \frac{3}{20}\,\epsilon_1\epsilon_2\epsilon_3\right) + \nu^2\,\frac{1}{30}\,\epsilon_2\epsilon_3$$
$$+ M\left\{\frac{7}{90}\,\epsilon_1\epsilon_2{}^2 - \frac{19}{60}\,\epsilon_1\epsilon_2\epsilon_3 - \nu\left(\frac{7}{144}\,\epsilon_2{}^2 - \frac{1}{8}\,\epsilon_2\epsilon_3\right)\right\}$$
$$- M^2\left(\frac{49}{360}\,\epsilon_2{}^2 - \frac{3}{20}\,\epsilon_2\epsilon_3\right).$$

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$V_3$ :

$$(v30) = \frac{1}{4} \epsilon_2 \epsilon_3^2,$$

$$(v31) = -\frac{1}{20} \epsilon_1 \epsilon_2^3 - \frac{2}{5} \epsilon_1 \epsilon_2^2 \epsilon_3 - \frac{3}{10} \epsilon_1 \epsilon_2 \epsilon_3^2 + v \frac{3}{20} \epsilon_2 \epsilon_3^2$$
$$-M \left( \frac{1}{30} \epsilon_2^3 + \frac{13}{45} \epsilon_2^2 \epsilon_3 - \frac{7}{20} \epsilon_2 \epsilon_3 \right).$$

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$V_4$ :

$$(v40) = \frac{1}{5} \epsilon_2 \epsilon_3^3.$$

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$X_0$ :

$$(x00) = -M \frac{1}{4} \epsilon_1,$$

$$(x01) = M \left( \frac{1}{9} \epsilon_1^2 + v \frac{2}{9} \epsilon_1 \right),$$

$$(x02) = -M \left( \frac{11}{192} \epsilon_1^3 + v \frac{7}{48} \epsilon_1^2 + v^2 \frac{3}{16} \epsilon_1 \right),$$

$$(x03) = M \left( \frac{19}{600} \epsilon_1^4 + v \frac{1}{10} \epsilon_1^3 + v^2 \frac{23}{150} \epsilon_1^2 + v^3 \frac{4}{25} \epsilon_1 \right).$$

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$X_1$ :

$$(x10) = M \left( \frac{3}{4} \epsilon_2 - \frac{1}{2} \epsilon_3 \right),$$

$$(x11) = -M \left( \frac{7}{18} \epsilon_1 \epsilon_2 - \frac{1}{6} \epsilon_1 \epsilon_3 - v \frac{1}{18} \epsilon_2 \right) + M^2 \left( \frac{11}{36} \epsilon_2 - \frac{1}{6} \epsilon_3 \right),$$

$$(x12) = M \left\{ \frac{25}{96} \epsilon_1^2 \epsilon_2 - \frac{1}{12} \epsilon_1^2 \epsilon_3 \right.$$
$$\left. + v \left( \frac{17}{144} \epsilon_1 \epsilon_2 - \frac{1}{12} \epsilon_1 \epsilon_3 \right) - v^2 \frac{1}{72} \epsilon_2 \right\}$$
$$- M^2 \left\{ \frac{47}{576} \epsilon_1 \epsilon_2 + \frac{1}{48} \epsilon_1 \epsilon_3 + v \left( \frac{5}{96} \epsilon_2 - \frac{1}{24} \epsilon_3 \right) \right\}$$
$$+ M^3 \left( \frac{25}{288} \epsilon_2 - \frac{1}{24} \epsilon_3 \right),$$

$$(x13) = -\,\mathrm{M}\left\{\frac{43}{240}\,\epsilon_1{}^3\epsilon_2 - \frac{11}{240}\,\epsilon_1{}^3\epsilon_3 + \nu\left(\frac{227}{1200}\,\epsilon_1{}^2\epsilon_2 - \frac{1}{12}\,\epsilon_1{}^2\epsilon_3\right)\right.$$
$$\left. + \nu^2\left(\frac{109}{1800}\,\epsilon_1\epsilon_2 - \frac{1}{20}\,\epsilon_1\epsilon_3\right) - \nu^3\,\frac{1}{180}\,\epsilon_2\right\}$$
$$+ \mathrm{M}^2\left\{\frac{77}{2400}\,\epsilon_1{}^2\epsilon_2 + \frac{11}{360}\,\epsilon_1{}^2\epsilon_3 + \nu\left(\frac{3}{800}\,\epsilon_1\epsilon_2 + \frac{13}{360}\,\epsilon_1\epsilon_3\right)\right.$$
$$\left. + \nu^2\left(\frac{59}{3600}\,\epsilon_2 - \frac{1}{60}\,\epsilon_3\right)\right\}$$
$$- \mathrm{M}^3\left\{\frac{391}{14400}\,\epsilon_1\epsilon_2 + \frac{1}{240}\,\epsilon_1\epsilon_3 + \nu\left(\frac{13}{288}\,\epsilon_2 - \frac{1}{40}\,\epsilon_3\right)\right\}$$
$$+ \mathrm{M}^4\left(\frac{137}{7200}\,\epsilon_2 - \frac{1}{120}\,\epsilon_3\right).$$

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$\mathrm{X}_2:$

$$(x20) = \mathrm{M}\left(\frac{1}{6}\,\epsilon_2{}^2 - \frac{1}{18}\,\epsilon_2\epsilon_3\right),$$

$$(x21) = -\,\mathrm{M}\left\{\frac{47}{192}\,\epsilon_1\epsilon_2{}^2 - \frac{1}{16}\,\epsilon_1\epsilon_2\epsilon_3 - \nu\left(\frac{1}{12}\,\epsilon_2{}^2 - \frac{1}{72}\,\epsilon_2\epsilon_3\right)\right\}$$
$$- \mathrm{M}^2\left(\frac{7}{64}\,\epsilon_2{}^2 - \frac{91}{288}\,\epsilon_2\epsilon_3 + \frac{1}{8}\,\epsilon_3{}^2\right),$$

$$(x22) = \mathrm{M}\left\{\frac{4}{15}\,\epsilon_1{}^2\epsilon_2{}^2 - \frac{11}{200}\,\epsilon_1{}^2\epsilon_2\epsilon_3\right.$$
$$\left. - \nu\left(\frac{59}{1200}\,\epsilon_1\epsilon_2{}^2 + \frac{1}{75}\,\epsilon_1\epsilon_2\epsilon_3\right) + \nu^2\,\frac{1}{150}\,\epsilon_2\epsilon_3\right\}$$
$$+ \mathrm{M}^2\left\{\frac{139}{2400}\,\epsilon_1\epsilon_2{}^2 - \frac{943}{3600}\,\epsilon_1\epsilon_2\epsilon_3 + \frac{1}{12}\,\epsilon_1\epsilon_3{}^2\right.$$
$$\left. + \nu\left(\frac{1}{48}\,\epsilon_2{}^2 + \frac{1}{45}\,\epsilon_2\epsilon_3\right)\right\}$$
$$- \mathrm{M}^3\left(\frac{99}{800}\,\epsilon_2{}^2 - \frac{97}{400}\,\epsilon_2\epsilon_3 + \frac{1}{12}\,\epsilon_3{}^2\right).$$

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$\mathrm{X}_3:$

$$(x30) = \mathrm{M}\left(\frac{1}{12}\,\epsilon_2{}^2\epsilon_3 - \frac{1}{48}\,\epsilon_2\epsilon_3{}^2\right),$$

$$(x31) = -M\left\{ \frac{29}{400}\,\epsilon_1\epsilon_2{}^3 + \frac{121}{1200}\,\epsilon_1\epsilon_2{}^2\epsilon_3 - \frac{7}{300}\,\epsilon_1\epsilon_2\epsilon_3{}^2 \right.$$
$$\left. -\nu\left(\frac{1}{15}\,\epsilon_2{}^2\epsilon_3 - \frac{7}{600}\,\epsilon_2\epsilon_3{}^2\right)\right\}$$
$$-M^2\left(\frac{61}{400}\,\epsilon_2{}^3 - \frac{103}{400}\,\epsilon_2{}^2\epsilon_3 + \frac{223}{3600}\,\epsilon_2\epsilon_3{}^2\right).$$

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$X_4$ :
$$(x40) = M\left(\frac{1}{20}\,\epsilon_2{}^2\epsilon_3{}^2 - \frac{1}{100}\,\epsilon_2\epsilon_3{}^3\right).$$

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$P_1$ :
$$(p10) = 1,$$
$$(p11) = -\epsilon_1,$$
$$(p12) = \epsilon_1{}^2 + M\frac{1}{4}\,\epsilon_1,$$
$$(p13) = -\frac{11}{12}\,\epsilon_1{}^3 - \nu\frac{1}{3}\,\epsilon_1{}^2 - M\left(\frac{13}{36}\,\epsilon_1{}^2 + \nu\frac{2}{9}\,\epsilon_1\right).$$

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$P_2$ :
$$(p20) = \epsilon_3,$$
$$(p21) = -\epsilon_1\epsilon_2 - \epsilon_1\epsilon_3 - M\left(\frac{3}{4}\,\epsilon_2 - \frac{1}{2}\,\epsilon_3\right),$$
$$(p22) = \frac{11}{6}\,\epsilon_1{}^2\epsilon_2 + \epsilon_1{}^2\epsilon_3 - \nu\frac{1}{3}\,\epsilon_1\epsilon_2$$
$$+ M\left(\frac{29}{36}\,\epsilon_1\epsilon_2 - \frac{1}{6}\,\epsilon_1\epsilon_3 - \nu\frac{1}{18}\,\epsilon_2\right) - M^2\left(\frac{11}{36}\,\epsilon_2 - \frac{1}{6}\,\epsilon_3\right).$$

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$P_3$ :
$$(p30) = \epsilon_3{}^2,$$
$$(p31) = -\frac{1}{4}\,\epsilon_1\epsilon_2{}^2 - \frac{5}{3}\,\epsilon_1\epsilon_2\epsilon_3 - \epsilon_1\epsilon_3{}^2 - M\left(\frac{1}{6}\,\epsilon_2{}^2 + \frac{13}{9}\,\epsilon_2\epsilon_3 - \epsilon_3{}^2\right).$$

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$P_4$ :
$$(p40) = \epsilon_3{}^3.$$

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The higher coefficients for each subsidiary series can be computed following the procedure outlined ; but in practice this is not necessary, as, to account for the residue terms in the subsidiary series, it suffices to weight the last term retained by the quantity $(0, 1)$. If M be small the subsidiary series converge rapidly, while if M be large * the leading terms converge rapidly, as may be seen by considering the first term of the $X_1Z^4$ series, in the expansion for $\log ZX$, namely,

$$M\left(\frac{1}{20}\epsilon_2{}^2\epsilon_3{}^2 - \frac{1}{100}\epsilon_2\epsilon_3{}^3\right)\left(5, \frac{4M}{\nu} + 3\right),$$

which is the order of a fraction of $Me^{-4M}$ for $\nu = 0$, and a fraction of $M \cdot 2^{-(8M+3)}$ for $\nu = \frac{1}{2}$, and for $\lim z = 1$. The coefficients for the P series have been shown to $P_4$, as they are required in the evaluations to the extent developed ; the calculation of P is, however, best effected by substituting for V and X in equation (20). The explicit forms of the coefficients indicate the manner in which the subsidiary series could be developed in ascending powers of $\nu$, with functions of $(1, 0)$ or of $(1, -1)$ as weighting values.

## V. Methods of Approximation and Supplementary Analysis.

19. *Approximate Solution for* V.—The preceding methods can be utilized to obtain the functions V, X, P to any degree of accuracy required ; a procedure, much shorter and numerically sufficiently accurate, will now be outlined to investigate to a first order the circumstances of any particular assembly of conditions. With this end in view advantage is taken of the form of equation (13), which involves a direct relationship between the three functions V, X, P without the explicit occurrence of the variable $z$.

It has been shown previously that equating V to $z$ is not merely an analytical approximation to a solution for V in terms of $z$, but in most cases is numerically not far removed from the truth by reason of the compensating influences of the $\epsilon_1$ and $\epsilon_2$ terms in the equation

$$\frac{dV}{dz} = 1 - \frac{\epsilon_1 V^2}{z(1 - \nu z)} + \epsilon_2 P. \quad . \quad . \quad . \quad (18)$$

For the smaller values of M, V numerically is slightly in excess of $z$, as $z$ tends to 1 ; as M increases V numerically becomes slightly less than $z$, for $z$ tending to 1.

* A large value of M in practice is of the order of 2.

For present purposes therefore the solution of equation (18) may be taken to be

$$V = z + \epsilon_1 \left\{ \frac{1}{\nu^2} \log(1-\nu z) + \frac{z}{\nu} \right\} - \epsilon_2 \frac{z(1-\nu z)^{\frac{M}{\nu}+2}}{M+2\nu}$$
$$+ \frac{\epsilon_2}{(M+2\nu)(M+3\nu)} \left\{ 1 - (1-\nu z)^{\frac{M}{\nu}+3} \right\}, \quad (62)$$

the form being derived as a second approximation by writing in the small order terms on the right-hand side as a first approximation

$$V = z \quad \text{and} \quad P = z(1-\nu z)^{\frac{M}{\nu}+1}.$$

In consequence V may be regarded as a known quantity *.

20. *Modification of the* P *Function Form.*—A further equation is required in V, X, P not explicitly involving $z$, so that one of the functions X and P may be eliminated between equation (13) and the additional equation, the derived relationship giving the retained function and its differential coefficient with respect to V in terms of V. The solution of this derived equation gives in terms of V the value of the retained function, and consequently also the value of the eliminated function.

Without serious error equation (6) may be written

$$P = \frac{z(1-\nu z) - \epsilon_1 V^2}{X \left\{ 1 - \epsilon_3 z (1-\nu z)^{\frac{M}{\nu}+1} \right\}}, \quad \cdot \quad \cdot \quad \cdot \quad (63)$$

in which the first approximation to the value of X has been written in the small order term involving $\epsilon_3$.

It has been shown that V may be expanded as a series in powers of $z$, and that analytically and numerically $V = z$ is a close approximation. We may in consequence write

$$V = z + n V^2,$$

where $n$ is a small order quantity, positive for M small, passing through zero for an intermediate value of M, and becoming negative and very small for larger values of M. It follows that a close approximation to equation (63 is)

$$P = \frac{V(1-\nu V) - \epsilon_1 V^2}{X}, \quad \cdot \quad \cdot \quad \cdot \quad (64)$$

which may be seen by consideration of the following extreme cases.

* Although the parameter $\epsilon_3$ does not occur explicitly in equation (62) it physically plays a part through its occurrence in the expression for X, V being correlated with X through $z$.

I. M *small.*—Here $n$ is positive, and substituting for $z$ in equation (63)

$$P = \frac{\{V(1-\nu V)-\epsilon_1 V^2\}-\{nV^2(1-2\nu V)+n^2\nu V^4\}}{X\{1-\epsilon_3 V(1-\nu V)^{\frac{M}{\nu}+1}\}}. \quad (65)$$

For $\nu=\frac{1}{2}$ the expression $nV^2(1-2\nu V)$ is always positive and small for small values of V, and also for $V \longrightarrow 1$, the intermediate values also being small since $n$ is small. The expression $n^2\nu V^4$ is positive and small. Hence the term $\{nV^2(1-2\nu V)+n^2\nu V^4\}$ in the numerator and the term $\epsilon_3 V(1-\nu V)^{\frac{M}{\nu}+1}$ in the denominator always carry the same sign, and as they are respectively small fractions of the major terms in the expression, very nearly the same order of magnitude, their neglect will introduce compensating numerical errors.

For $\nu=0$ the expression becomes

$$P = \frac{V-\epsilon_1 V^2-nV^2}{X\{1-\epsilon_3 V e^{-MV}\}}, \quad . \quad . \quad . \quad (66)$$

and again the neglect of the small order terms will introduce numerical errors tending to compensate.

II. M *larger.*—Here $n$ is negative, and for $\nu=\frac{1}{2}$ the term $\{nV^2(1-2\nu V)+n^2\nu V^4\}$ may be positive or negative, according to the value of V. It is negative for V small, and ultimately may become positive for $V \longrightarrow 1$, since in this region $(1-2\nu V) \longrightarrow 0$ and $n^2\nu V^4$ is always positive. For the values of M now under consideration $\epsilon_3 V(1-\nu V)^{\frac{M}{\nu}+1}$ becomes very small compared with 1 as $V \longrightarrow 1$. Both subsidiary terms are therefore small and may be neglected without serious numerical error. For $\nu=0$ the expression takes the same form algebraically as equation (66), but the subsidiary terms in the numerator and denominator are now of opposite sign. Their neglect will in consequence result in a value of P numerically slightly in defect of the value that would derive from equation (63). The error, however, is not large, as in this special case both quantities neglected are very small in comparison with the main terms [*].

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[*] The case of a large M together with $\nu=0$ is rarely met with in practice.

A closer approximation to equation (6) may be reached by writing

$$P = \frac{V(1-\nu V) - \epsilon V_1^2}{X}, \quad \ldots \quad (67)$$

which is derived from equation (64) by the substitution of $\epsilon$ for $\epsilon_1$ where $\epsilon$ is a small quantity at choice. Since the expression conserves the principal analytical features of equation (6), whatever value X may take, and since $\epsilon$ is small, the form in great measure reproduces the curvature of P regarded as a function of $z$. Its values numerically adhere closely to the exact P values when $z$ is small and gradually depart from them for $z \longrightarrow 1$, being in some cases in this region slightly in excess and in others, *e.g.*, M large, $\nu = 0$, slightly in defect.

If, now, $\epsilon$ can *à priori* be chosen, so that equation (67) leads to numerical agreement with the exact value of P in the region of $z$ where departure is likely to be most pronounced, namely, $z = 1$, and if it be found that $\epsilon$ as so prescribed conforms in order of magnitude to the orders of magnitude of $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, then equation (67) with the specially selected value of $\epsilon$ is substantially the equivalent of the exact form of P both analytically and numerically, and may be used to replace it with no material loss. Further, since with the form of P so derived X is to be evaluated by integration, the error in X regarded as a function of $z$ is effectively negligible, as also the error in P evaluated from the derived value of X.

To ascertain the required value of $\epsilon$ it suffices to write

$$V_B(1-\nu V_B) - \epsilon V_B^2 = \frac{1-\nu-\epsilon_1 V_B^2}{1-\epsilon_3(1-\nu)^{\frac{M}{\nu}+1}}, \quad . \quad (68)$$

where $V_B$ is the value of V for $z = 1$ from equation (62). This may be solved immediately or by neglecting expressions containing powers and products of $\epsilon_1$, $\epsilon_2$, $\epsilon_3$, since the quantities are small.

Adopting the latter mode of solution,

$$\epsilon = \epsilon_1 \left\{ 1 + (1-2\nu) \left( \frac{1}{\nu^2} \log(1-\nu) + \frac{1}{\nu} \right) \right\}$$
$$+ \epsilon_2 \frac{1-2\nu}{(M+2\nu)(M+3\nu)} \left\{ 1 - (1-\nu)^{\frac{M}{\nu}+3} \right.$$
$$\left. - (M+3\nu)(1-\nu)^{\frac{M}{\nu}+2} \right\} - \epsilon_3(1-\nu)^{\frac{M}{\nu}+2}, \quad (69)$$

whence it follows that $\epsilon$ as prescribed is small and comparable in order of magnitude with $\epsilon_1$, $\epsilon_2$, $\epsilon_3$.

For $\nu = \frac{1}{2}$ equation (69) reduces to

$$\epsilon = \epsilon_1 - \epsilon_3 2^{-(2M+2)},$$

so that in general for this case $\epsilon$ is very nearly equal to $\epsilon_1$.

For $\nu = 0$ the relationship becomes

$$\epsilon = \tfrac{1}{2}\epsilon_1 + \epsilon_2 \frac{1}{M^2}(1 - e^{-M} - M e^{-M}) - \epsilon_3 e^{-M}.$$

Here the coefficient of $\epsilon_2$ is always positive, since $e^M > (1 + M)$. Further, the expression

$$\epsilon_2 \frac{1}{M^2}(1 - e^{-M} - M e^{-M}) - \epsilon_3 e^{-M}$$

is positive provided that

$$e^M > 1 + M + \frac{\epsilon_3}{\epsilon_2} M^2 > 1 + M + \frac{\eta - \dfrac{1}{\delta}}{\eta} M^2$$

*i. e.*, provided that $\eta < \frac{2}{\delta}$, a condition that obtains for the generality of propellant compositions. Hence in general $\epsilon$ is positive for $\nu = 0$.

21. *Solution of the Equations.*—The scheme of equations required is

$$MV \frac{dV}{dX} = P, \qquad \cdots \qquad (13)$$

$$P = \frac{V\{1 - (\nu + \epsilon)V\}}{X}, \qquad \cdots \qquad (67)$$

$$V = z + \epsilon_1 \left\{ \frac{1}{\nu^2} \log(1 - \nu z) + \frac{z}{\nu} \right\} - \epsilon_2 \frac{z(1 - \nu z)^{\frac{M}{\nu}+2}}{M + 2\nu}$$

$$+ \frac{\epsilon_2}{(M + 2\nu)(M + 3\nu)}\{1 - (1 - \nu z)^{\frac{M}{\nu}+3}\}. \qquad (62)$$

Eliminating P between equations (13) and (67), and integrating,

$$X = \{1 - (\nu + \epsilon)V\}^{-\frac{M}{\nu + \epsilon}},$$

and it follows that

$$P = V\{1 - (\nu + \epsilon)V\}^{\frac{M}{\nu + \epsilon}+1}. \qquad \cdots \qquad (71)$$

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22. *Conditions at Maximum Pressure.*—The ascertainment of the maximum value of P during the combustion is an essential requirement. The numerical procedure outlined in Section II. does not enable any explicit expression to be formulated for $P_{max.}$, and the regional conditions can be obtained only by inspection of tabulated values. For convenience of presentation the establishment of a maximum pressure formula by the foregoing approximate method will be considered first, the subsequent application to the methods of Sections III. and IV. being thereby facilitated.

23. *Approximate Solution for* $P_{max.}$—Maximum pressure may be attained either during combustion of the charge, *i. e.*, for a value of $z < 1$, the pressure subsequently falling as $z \longrightarrow 1$, or at the end of combustion, the pressure rising steadily with $z$ increasing up to the value $z = 1$. If the first condition obtain it must result from $dP/dz = 0$ for $z < 1$, while for the second case $dP/dz$ cannot vanish for any value of $z$ between 0 and 1.

From equation (71), the condition that $P_{max.}$ may occur for $z < 1$ requires

$$\frac{1}{P_{max.}} \frac{dP}{dV} = 0 = \frac{1}{V_{max.}} - \frac{M + \nu + \epsilon}{1 - (\nu + \epsilon) V_{max.}},$$

whence

$$V_{max.} = \frac{1}{M + 2\nu + 2\epsilon}.$$

This entails the condition $(M + 2\nu + 2\epsilon) > V_B$, giving a critical value of M.

Assuming the criterion to hold :

$$P_{max.} = \frac{(M + \nu + \epsilon)^{\frac{M+\nu+\epsilon}{\nu+\epsilon}}}{(M + 2\nu + 2\epsilon)^{\frac{M+2\nu+2\epsilon}{\nu+\epsilon}}}, \quad \cdot \quad \cdot \quad \cdot \quad (72)$$

whence, writing $y = \nu + \epsilon$, where $\left| \dfrac{y}{M+y} \right| < 1$,

$$-\log P_{max.} = \log (M + 2y) + \frac{M+y}{y} \log \left(1 + \frac{y}{M+y}\right),$$

whence

$$\frac{1}{P_{max.}} = (M + 2y) e^{\left\{1 - \frac{1}{2}\frac{y}{M+y} + \frac{1}{3}\left(\frac{y}{M+y}\right)^2 - \frac{1}{4}\left(\frac{y}{M+y}\right)^3 + \dots\right\}}$$

$$= e \left\{ M + \frac{3}{2}y - \frac{1}{24}y\frac{y}{M+y} + \frac{1}{48}y\left(\frac{y}{M+y}\right)^2 - \dots \right\},$$

in the right-hand side of which only the first two terms need to be retained, so that

$$P_{max.} = \frac{1}{e\{M + \frac{3}{2}(\nu + \epsilon)\}}. \quad . \quad . \quad . \quad (73)$$

If the criterion $(M + 2\nu + 2\epsilon) > V_B$ does not hold, then the equation for $P_{max.}$ is

$$P_{max.} = V_B\{1 - (\nu + \epsilon)V_B\}^{\frac{M}{\nu + \epsilon} + 1},$$

which approximates closely to

$$P_{max.} = \{1 - (\nu + \epsilon)\}^{\frac{M}{\nu + \epsilon} + 1}. \quad . \quad . \quad . \quad (74)$$

24. *Solution for $P_{max.}$ by the Methods of Sections III. and IV.*—From equation (6) the condition for maximum pressure before the end of combustion is given by

$$0 = \frac{1}{P_{max.}}\frac{dP}{dz} = \frac{(1 - 2\nu z) - 2\epsilon_1 V \dfrac{dV}{dz}}{z(1 - \nu z) - \epsilon_1 V^2} - \frac{\dfrac{dX}{dz} - \epsilon_3(1 - 2\nu z)}{X - \epsilon_3 z(1 - \nu z)}.$$

It follows, using equation (13), that

$$MV\frac{dV}{dz} - \epsilon_3(1 - 2\nu z)P_{max.} = (1 - 2\nu z) - 2\epsilon_1 V\frac{dV}{dz},$$

from which the value of $z$ for $P_{max.}$ has to be ascertained. It suffices to write

$$V = z,$$

$$\frac{dV}{dz} = 1 - \epsilon_1\frac{z}{1 - \nu z} + \epsilon_2 P_{max.},$$

leading to

$$Mz\left(1 - \epsilon_1\frac{z}{1 - \nu z} + \epsilon_2 P_{max.}\right) - \epsilon_3(1 - 2\nu z)P_{max.}$$

$$= (1 - 2\nu z) - 2\epsilon_1 z\left(1 - \epsilon_1\frac{z}{1 - \nu z} + \epsilon_2 P_{max.}\right). \quad (75)$$

The first approximation to $z_{max.}$ is $1/(M + 2\nu + 2\epsilon_1)$, from which, by the method of the preceding paragraph, a near value for $P_{max.}$ is given by

$$P_{max.} = \frac{1}{\epsilon\{M + \frac{3}{2}(\nu + \epsilon_1)\}}.$$

Substituting these approximate values for $z_{max.}$ and $P_{max.}$ in

the small order terms of equation (75), a second approximation to $z_{max.}$ is given by

$$z_{max.} = \frac{1}{M + 2(\nu + \epsilon_1)}$$
$$1 + \frac{\dfrac{\epsilon_3(M + 2\epsilon_1)}{e\{M + \frac{3}{2}(\nu + \epsilon_1)\}\{M + 2(\nu + \epsilon_1)\}}}{\left\{1 - \dfrac{\epsilon_1(M + 2\epsilon_1)}{(M + \nu + 2\epsilon_1)\{M + 2(\nu + \epsilon_1)\}} + \dfrac{\epsilon_2(M + 2\epsilon_1)}{\{M + 2(\nu + \epsilon_1)\}e\{M + \frac{3}{2}(\nu + \epsilon_1)\}}\right\}}.$$
$$. \quad . \quad . \quad (76)$$

The value of $z_{max.}$ given by equation (76) requires to be substituted in the appropriate expressions for V and X obtained in Sections III. and IV., whence, using equation (6), the value of $P_{max.}$ may be determined.

25. *Shot Travel to* $P_{max.}$—Using the approximate method the value of X corresponding to $P_{max.}$ follows from equation (70).

Substituting therein the value of $V_{max.}$ previously obtained,

$$X_{max.} = \left\{\frac{M + 2(\nu + \epsilon)}{M + \nu + \epsilon}\right\}^{\frac{M}{\nu + \epsilon}},$$

and using equation (73),

$$X_{max.} = \frac{e(M + \nu + \epsilon)\{M + \frac{3}{2}(\nu + \epsilon)\}}{\{M + 2(\nu + \epsilon)\}^2}. \quad . \quad . \quad (77)$$

For the methods of Sections III. and IV. the value of $z_{max.}$ from equation (76) must be substituted in the general expressions for X.

26. *Conditions at Complete Combustion.*—An approximation to the value of $V_B$ is given by writing $z = 1$ in equation (62), whence

$$V_B = 1 + \epsilon_1\left\{\frac{1}{\nu^2}\log(1 - \nu) + \frac{1}{\nu}\right\}$$
$$+ \frac{\epsilon_2}{(M + 2\nu)(M + 3\nu)}\left\{1 - (M + 2\nu + 1)(1 - \nu)^{\frac{M}{\nu} + 2}\right\}, (78)$$

and from equation (70)

$$X_B = \{1 - (\nu + \epsilon)V_B\}^{-\frac{M}{\nu + \epsilon}}; \quad . \quad . \quad . \quad (79)$$

also from equation (71)

$$P_{B} = V_{B}\{1 - (\nu + \epsilon)V^{B}\}^{\frac{M + \nu + \epsilon}{\nu + \epsilon}} . \quad . \quad . \quad . \quad (80)$$

For the methods of Sections III. and IV. $V_{B}$, $X_{B}$, $P_{B}$ are obtained by writing $z = 1$ in the appropriate general expressions for V, X, P.

27. *Motion after Complete Combustion for Short Additional Shot Travel* \*.—From this point onwards the scheme of equations in paragraph (6) reduces to

$$\tau = 1 - \frac{\epsilon_1 V^2}{1 - \nu}, \quad . \quad . \quad . \quad . \quad . \quad (81)$$

$$MV\frac{dV}{dX} = P, \quad . \quad . \quad . \quad . \quad (13)$$

$$P = \frac{1 - \nu - \epsilon_1 V^2}{X - \epsilon_3(1 - \nu)}. \quad . \quad . \quad . \quad (82)$$

From equations (13) and (82)

$$\frac{MVdV}{1 - \nu - \epsilon_1 V^2} = \frac{dX}{X - \epsilon_3(1 - \nu)},$$

whence, integrating and using the conditions that at the instant of complete combustion of the propellant charge $V = V_{B}$, $X = X_{B}$,

$$V^2 = V_{B}^2 \left(\frac{X_{B} - \epsilon_3(1 - \nu)}{X - \epsilon_3(1 - \nu)}\right)^{\frac{2\epsilon_1}{M}} + \frac{1 - \nu}{\epsilon_1}\left\{1 - \left(\frac{X_{B} - \epsilon_3(1 - \nu)}{X - \epsilon_3(1 - \nu)}\right)^{\frac{2\epsilon_1}{M}}\right\}, \quad (83)$$

giving V for any $X > X_{B}$.

Substituting for V in equation (82), there follows

$$P = \frac{(1 - \nu - \epsilon_1 V_{B}^2)\{X_{B} - \epsilon_3(1 - \nu)\}^{\frac{2\epsilon_1}{M}}}{\{X - \epsilon_3(1 - \nu)\}^{1 + \frac{2\epsilon_1}{M}}}, \quad . \quad (84)$$

and from equation (81)

$$\tau = \left(1 - \frac{\epsilon_1 V_{B}^2}{1 - \nu}\right)\left(\frac{X_{B} - \epsilon_3(1 - \nu)}{X - \epsilon_3(1 - \nu)}\right)^{\frac{2\epsilon_1}{M}}. \quad . \quad . \quad (85)$$

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\* For prolonged additional shot travel equations (81) and (82) require modification in $\epsilon_1$, as a mean average value cannot be assumed operative for $s$.

## Summary.

The aim of the paper has been the establishment of a system of internal ballistics, based on the assumption of the particular forms we have recently put forward for the equation of state of propellant gases and for the rate of burning law of stabilized colloidal propellants. The main analysis has centred round the conditions obtaining during the propellant combustion phase, and these are chiefly discussed in Sections III. and IV. An abridged procedure has been outlined in Section V. together with a brief reference to particular features of the motion both during and immediately subsequent to propellant combustion. The analysis has been specifically restricted to supposedly known conditions, no attempt being made to deal with wider applications, an adequate discussion of which would compel as a preliminary the introduction and detailed examination of several essential criteria not required for the immediate purposes in view.

In conclusion, we desire to express our thanks to Captain A. C. Goolden, R.N., for his encouragement and interest in the work, and to acknowledge our indebtedness to the Director of Artillery, the Director of Naval Ordnance, and the Ordnance Committee for their courtesy in sanctioning publication.

LIX. *The Electromagnetic Field of a moving uniformly and rigidly Electrified Sphere and its Radiationless Orbits.* By G. A. SCHOTT. *F.R.S., University College of Wales, Aberystwyth* *.

1. IT is very generally assumed that an electrified body in accelerated motion necessarily loses energy by radiation at a rate proportional to the square of its acceleration. In the course of an investigation carried out recently, I have discovered a case in which this assumption is certainly false on the basis of the Electromagnetic Theory of Maxwell and its further developments by Larmor and H. A. Lorentz. In order to make quite clear what is involved let us consider a concrete example : imagine a metal sphere suspended

by a fine metal wire in such a manner that it can be earthed or insulated at will. Surround it by a closely fitting insulating coating. *e. g.*, two thin hollow hemispheres of ebonite fitted together, and then place around the who'e and concentric with it a larger insulated metal sphere made of two hemispheres with a very small hole through which the suspending wire passes without touching the outer sphere. Now connect the outer metal sphere to one pole of a battery and the inner one momentarily to the other pole, and again insulate the latter sphere. This receives a charge which resides on the ebonite in contact with it according to the theory of Maxwell and a well known experiment of Faraday. Remove the outer metal hemispheres and also the ebonite hemispheres. Joining the latter together we obtain a very nearly uniformly charged insulating sphere, and, if the ebonite insulated perfectly, the charge would remain uniform however the sphere moved about as a whole. This charged ebonite sphere is a concrete example, realized approximately, of what is meant in the present paper. It will be shown that, if the centre of such a uniformly and rigidly electrified sphere describes a closed orbit of any form with a suitably chosen period, and the sphere rotates with such an angular velocity that every point of it describes an equal and parallel orbit, then the electromagnetic field due to the sphere at a sufficient distance is a static field, and therefore no energy will be radiated to infinity. The motion of the sphere must be one of pure translation without any spin.

2. In order to prove this result we start from the following expressions for the scalar potential $\phi$ and vector potential $\mathbf{a}$ *

$$\phi = \frac{1}{2\pi} \int de \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mu(t-\tau-\mathrm{R})} d\tau \, d\mu / \mathrm{R} \quad . \quad . \quad (1)$$

$$\mathbf{a} = \frac{1}{2\pi} \int de \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\mu(t-\tau-\mathrm{R})} \mathbf{v} d\tau \, d\mu / \mathrm{R}, \quad . \quad . \quad (2)$$

* Schott, *Ann. der Physik*, xxvi. p. 637 (1907); 'Electromagnetic Radiation,' ch. II. §9, eqs. (12) and (13).

where **v** is the common vector velocity of every element $de$ of the charge on the sphere, and R is the distance of the field point from $de$, both at the time $\tau$. To simplify printing and writing the unit of time has been taken to be the time taken by an electromagnetic disturbance to travel unit distance. To return to the usual unit we replace $t$ and $\tau$ by $ct$ and $c\tau$, and **v** by **v**/$c$.

We assume that the integration with respect to $de$ can be interchanged with those with respect to $t$ and $\mu$; the validity of this change of order of integration, as well as the character of the integrals (1) and (2), is discussed elsewhere.

Let the total charge on the sphere be $e$, the radius $a$, the distance of the field-point from the centre at time $\tau$, $r$, and the angle between r and the radius to the element $de$, $\gamma$. Since there is symmetry for the sphere and its charge about r, we may write

$$R^2 = a^2 + r^2 - 2ar\cos\gamma, \ \ de = \tfrac{1}{2}e\sin\gamma d\gamma = eRdR/2ar . \quad . \quad (3)$$

Changing the order of integration in (1) and using (3) we get

$$\phi = \frac{e}{4\pi a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mu(t-\tau)}d\tau\,d\mu}{r}\int_{R_1}^{R_2}e^{\cdot i\mu R}dR$$

$$= \frac{e}{4\pi a}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{i\mu(t-\tau-R_1)} - e^{i\mu(t-\tau-R_2)}}{i\mu r}\,d\tau\,d\mu,$$

$$= \frac{e}{2\pi a}\int_{-\infty}^{\infty}\frac{d\tau}{r}\int_{0}^{\infty}\frac{\sin\mu(t-\tau-R_1) - \sin\mu(t-\tau-R_2)}{\mu}\,d\mu,$$

$$\qquad\qquad\qquad\qquad . \quad . \quad (4)$$

where $R_1$ and $R_2$ are the least and greatest distances of the field-point from the sphere, both taken positively. The last expression is derived by dividing the range of integration for $\mu$ into two, from $-\infty$ to 0 and from 0 to $\infty$, and changing the sign of $\mu$ in the first half. In the same way we obtain from (2) an expression for **a** differing from (4) only by an additional factor **v** in the integral with respect to $\tau$.

3. In order to make further progress we must consider the kinematics of the problem briefly. As the centre

of the sphere describes its orbit, whatever it be, the
sphere sweeps out a region of space bounded by the
envelope of the moving sphere ; this envelope can also
be generated by a great circle of the sphere, whose plane
always remains normal to the path of the common centre.
Two cases arise : in one the envelope does not intersect
itself, and consists of a single sheet of tubular form,
either extending to infinity or forming a closed ring ;
in the other it intersects itself and consists of two sheets,
an outer sheet generated by the outer part of the moving
great circle, and an inner sheet generated by the inner
part. Any field-point, outside the single sheet of the
envelope in the first case, or the outer sheet in the second,
is such that $r > a$ for every position of the moving sphere ;
any field-point inside the inner sheet of the envelope
in the second case is such that $r < a$ for every position
of the moving sphere. Every other field-point is such
that $r > a$ for some positions of the moving sphere,
and $r < a$ for others. The first two types of field-point
are comparatively easy to deal with, especially the
first ($r > a$ always) ; we shall call this type an outer
field-point, and restrict our investigation to it, because
it leads to general and interesting results.

4. *Outer field-point* : $r > a$, $R_1 = r - a$, $R_2 = r + a$.
The arguments of the two sines in (4) become

$$\mu(t + a - \tau - r) \quad \text{and} \quad \mu(t - a - \tau - r)$$

respectively, and the corresponding integrals with respect
to $\mu$ are discontinuous Dirichlet integrals, whose values
are

$$\pm \pi/2 \quad \text{for} \quad \tau + r \lessgtr t + a, \quad \text{or} \quad t - a,$$

as the case may be. They are easily reduced by means
of the substitution

$$\sigma = \tau + r, \ \frac{d\sigma}{d\tau} = 1 + \frac{dr}{d\tau} = K, \ d\tau = \frac{d\sigma}{K}, \quad . \quad . \quad (5)$$

where K is a Döppler factor and is positive so long as
the velocity $v$ of the sphere is less than $c$, the velocity
of light. We shall suppose that this is so, and then $\sigma$
increases continually from $-\infty$ to $\infty$ as $\tau$ increases
between the same limits. We obtain at once from (4)

$$\phi = \int_{t-a}^{t+a} \frac{e \, d\sigma}{2aKr}, \quad . \quad . \quad . \quad . \quad (6)$$

and in precisely the same way

$$\mathbf{a} = \int_{t-a}^{t+a} \frac{e\mathbf{v}d\sigma}{2a\mathrm{K}r}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7)$$

Returning to the usual units by replacing $t$ and $\sigma$ by $ct$ and $c\sigma$ and $\mathbf{v}$ by $v/c$ we obtain

$$\phi = \int_{t-a/c}^{t+a/c} \frac{ecd\sigma}{2a\mathrm{K}r}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (6')$$

$$\mathbf{a} = \int_{t-a/c}^{t+a/c} \frac{e\mathbf{v}d\sigma}{2a\mathrm{K}r}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (7')$$

where now we have

$$\sigma = \tau + \frac{r}{c}, \ \mathrm{K}\frac{d\tau}{d\tau} = 1 + \frac{dr}{cd\tau}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (5')$$

5. These expressions can be written in another more suggestive form. Let $\tau_1$ be the time at which a disturbance would have to be emitted from the centre in its position $Q_1$ in order to reach the field-point P at the time $t-a/c$, and let $\tau_2$, $Q_2$ correspond similarly to the time $t+a/c$, just as $\tau$ corresponds to $\sigma$. Also let

$$Q_1 P = r_1, \ Q_2 P = r_2 ;$$

then by (5')

$$\tau_1 + \frac{r_1}{c} = t - \frac{a}{c}, \quad \tau_2 + \frac{r_2}{c} = t + \frac{a}{c}. \quad \cdot \quad \cdot \quad (8)$$

Further, let $ds$ be the element of arc described by the centre in the interval from $\tau$ to $\tau + d\tau$, and $\mathbf{ds}$ the same element regarded as a vector. Then we have

$$\frac{cd\sigma}{\mathrm{K}} = cd\tau = \frac{cds}{v} = \frac{ds}{\beta}, \quad \frac{\mathbf{v}d\sigma}{\mathrm{K}} = \mathbf{v}d\tau = \mathbf{ds}, \ \beta = v/c$$

as usual. Then (6'), (7') and (8), give

$$\phi = \int_{Q_1}^{Q_2} \frac{eds}{2a\beta r}, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9)$$

$$\mathbf{a} = \int_{Q_1}^{Q_2} \frac{e\mathbf{ds}}{2ar} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (10)$$

In words, $\phi$ is the electrostatic potential at P due to a distribution of charge along the arc $Q_1 Q_2$ of the orbit of the centre of the sphere with line density $e/2a\beta$, whilst $\mathbf{a}$ is the vector potential of the same arc when it carries a uniform current of strength $e/2a$.

Since the positions $Q_1$, $Q_2$ of the centre of the sphere depend on the time $t$ as well as on the coordinates of the field-point P in accordance with (8), $\phi$ and **a** will both depend on the time in general. But, when the orbit of the centre is periodic, $\phi$ and **a** can become independent of the time, although they will still depend on the coordinates of the field-point and the nature of the orbit.

6. *Periodic orbit.*—The orbit is necessarily a closed curve fixed in position; let the time of one revolution be T. Obviously the values of $\phi$ and **a** will generally repeat themselves after any interval which is an integral multiple of T, for the end points $Q_1$, $Q_2$ of the arc concerned in (9) and (10) will be the same as they were originally. Moreover, we see from (8) that $Q_2$ will coincide with $Q_1$, whenever $2a$ is equal to $c$T, or an integral multiple $j$ thereof.

By considering this periodic orbit as the limit of $j$ ultimately coincident turns of a spiral, we deduce that, when $2a = jc$T, where $j$ is an integer,

$$\phi = j \oint \frac{eds}{2a\beta r} \qquad \cdots \qquad (11)$$

$$\mathbf{a} = j \oint \frac{e d\mathbf{s}}{2ar}, \qquad \cdots \qquad (12)$$

where each contour integral is taken once round the orbit.

We can confirm these results by deriving them directly from (6′) and (7′) by a very convenient, though not rigorous, symbolic method. For the sake of brevity write

$$F(t) = \int^t \frac{cd\sigma}{Kr} = D^{-1}\left[\frac{1}{Kr}\right],$$

where D denotes the operator $d/cdt$ with the usual unit of time. Then we obtain from (6′)

$$\phi = \frac{e}{2a}\left\{ F\left(t + \frac{a}{c}\right) - F\left(t - \frac{a}{c}\right) \right\} = \frac{e}{2a}(e^{aD} - e^{-aD})F(t)$$

$$= \frac{\sinh aD}{aD}\left[\frac{e}{Kr}\right] \qquad \cdots \qquad (13)$$

Similarly (7′) gives

$$\mathbf{a} = \frac{\sinh aD}{aD}\left[\frac{e\mathbf{v}}{cKr}\right] \qquad \cdots \qquad (14)$$

In words, we can obtain the retarded point potentials due to the uniformly and rigidly charged sphere, with charge $e$ and radius $a$, in purely translatory motion, by operating with $\sinh aD/aD$ on the corresponding retarded point potentials due to a charge $e$ concentrated at the centre.

Since the operator $\sinh aD/aD$ is an even function of $aD$, we can replace $aD$ by $i\omega a/c$, when the function operated on is a simple harmonic function of $\omega t$.

Now, when the motion of the centre is periodic with a period $T$, the point potentials can be expanded in Fourier series of the types

$$\left[\frac{e}{Kr}\right] = \sum_{n=0}^{\infty} \left( \Phi_n \cos \frac{2\pi nt}{T} + \Phi_n' \sin \frac{2\pi nt}{T} \right) \quad . \quad (15)$$

$$\left[\frac{e\mathbf{v}}{cKr}\right] = \sum_{n=0}^{\infty} \left( \mathbf{A}_n \cos \frac{2\pi nt}{T} + \mathbf{A}_n' \sin \frac{2\pi nt}{T} \right), \quad . \quad (16)$$

where the coefficients are functions of the coordinates of the field-point and of the parameters defining the orbit.

Then we obtain from (13) . . . (16)

$$\phi = \sum_{n=0}^{\infty} \frac{\sin (2\pi na/cT)}{2\pi na/cT} \left( \Phi_n \cos \frac{2\pi nt}{T} + \Phi_n' \sin \frac{2\pi nt}{T} \right) \quad (11)$$

$$\mathbf{a} = \sum_{n=0}^{\infty} \frac{\sin (2\pi na/cT)}{2\pi na/cT} \left( \mathbf{A}_n \cos \frac{2\pi nt}{T} + \mathbf{A}_n' \sin \frac{2\pi nt}{T} \right). \quad (18)$$

When $T = 2a/cj$, the factor outside the brackets in (17) and (18) reduces to $\sin \pi nj/\pi nj$; it vanishes when $j$ is an integer for all values of $n$ except zero, and then reduces to unity. Hence in this case we obtain

$$\phi = \Phi_0 = \frac{1}{T} \int_0^T \frac{e\,dt}{Kr} = \frac{jc}{2a} \int_0^T \frac{e\,d\tau}{r} = j \oint \frac{e\,ds}{2a\beta r},$$

$$\mathbf{a} = \mathbf{A}_0 = \frac{1}{T} \int_0^T \frac{e\mathbf{v}\,dt}{cKr} = \frac{j}{2a} \int_0^T \frac{e\mathbf{v}\,d\tau}{r} = j \oint \frac{e\,d\mathbf{s}}{2ar},$$

which are identical with (11) and (12).

7. Since the line density of the distribution along the orbit is $e/2a\beta$, the total charge for $j$ turns is

$$j \oint \frac{e\,ds}{2a\beta} = j \int_0^T \frac{ec\,d\tau}{2a} = \frac{jecT}{2a} = e,$$

as might have been expected. Moreover, the total current in the $j$ coincident turns is $je/2a$, when the orbit is only reckoned once. Hence we can express our result in the following words :—

When the centre of a uniformly and rigidly charged sphere, with charge $e$ and radius $a$, in purely translatory motion, describes a closed orbit periodically in a time $2a/cj$, where $j$ is any integer, the electromagnetic field at every outer point is a static field.

The electrostatic potential of this field is the same as that due to a charge $e$ distributed along the orbit of the centre with a linear density varying inversely as the velocity of the centre, which is the same as that of every point of the sphere, and the magnetic field is the same as that due to a uniform steady current of strength $je/2a$ flowing round the orbit of the centre in the direction of its motion.

At great distances from the orbit the electric force varies inversely as the square of the distance and the magnetic force inversely as its cube, so that the total flow of energy across a large sphere enclosing the orbit is zero in the limit as the radius approaches infinity, and the radiation from the moving sphere is zero, as usually defined, although there is acceleration.

This is true only when the motion of the sphere is one of pure translation ; spin always produces radiation.

It is noteworthy that the dimensions of these radiationless orbits are generally small compared with the diameter of the sphere. In fact, the perimeter $l$ of the orbit with period T is given by

$$l = \int_0^{\mathrm{T}} v\,d\tau = \bar{v}\mathrm{T} = 2a\bar{\beta}/j, \quad \ldots \quad (19)$$

where $\bar{\beta}$ is the time average of $v/c$ along the orbit. When this is less than unity, as we have supposed, the perimeter of the orbit is less than the diameter of the sphere and may be much less, if $\beta$ is small, or $j$ large. Thus the orbit of the centre lies entirely inside the sphere, in fact, inside the inner sheet of the envelope, which has two sheets.

8. *Application to problems of atomic structure.*—Having now established the principal result of our investigation, we may perhaps be permitted to indulge in a little

speculation, and, though models of the atom and its constituents, especially classical ones, are out of fashion, enquire whether such models, constructed out of charged spheres, like the one considered above, may not, after all, be of use in the elucidation of atomic problems. The chief difficulty has always been that stationary electric charges cannot form a stable system, whilst charges moving in closed orbits cannot be permanent in the presence of the radiation hitherto always supposed to accompany them ; but a charged sphere, such as ours, is not necessarily subject to the latter objection.    Obviously it does not help to account for Bohr's radiationless electron orbits, for, if one of our spheres were used as a model of the electron, the radiationless orbits of its centre would be far too small, since they would lie entirely inside the electron, as we saw in the last section.

But this very fact suggests that, if two of our spheres were taken as models of the electron and proton, it might prove possible to use them to construct a permanent model of the neutron, possibly also permanent models of atomic nuclei ; for we require radiationless orbits of nuclear dimensions, which can be provided by our spheres, if they are of such dimensions, and have no spin. In this connexion the experimental fact that the electron loses its spin, when it enters and exists inside the nucleus, is very suggestive.

Thus take a uniformly and rigidly charged sphere, with charge $e$ and radius $a$, as model of the proton, and a similar sphere, with charge $-e$ and radius $a'$, as model of the electron, both without any spin. Suppose that under their mutual attraction their centres describe periodic orbits about their common centroid in a periodic time T. From what we proved above it follows that they will not lose any energy by radiation if

$$\mathrm{T} = 2a/cj = 2a'/cj',$$

where $j$ and $j'$ are integers. For this to be possible $a$ and $a'$ must be commensurable and we must have

$$j' : j = a' : a = m : m',$$

apart from relativistic corrections, where $m$ and $m'$ are the masses of the proton and electron, which are approximately as $1850 : 1$. The average linear dimensions of the two orbits must be less than $a/\pi j$ and $a'/\pi j'$,

the greatest values being less than the radius of the proton, and therefore extremely small. Thus the accelerations are exceedingly great, but nevertheless there is no loss of energy due to radiation. The orbits and the proton itself are completely inside the electron—in fact, inside the inner sheet of the two-sheeted envelope of the electron. If under these conditions the mutual attraction proves to be such that a periodic motion of the type contemplated is possible, we shall have a stable and permanent model of the neutron, which could, however, be disrupted by disturbing forces powerful enough to pull the proton out of the electron. The study of such a model must be left for a future investigation ; if it should prove successful, we might be able to attack the problem of the structure of more complicated nuclei with some hope of success.

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## LX. *On the Definition of Distance in General Relativity.*
### *By* I. M. H. ETHERINGTON [*].

### § 1. *Introduction.*

IN recent papers Professor E. T. Whittaker [†] and H. S. Ruse [‡] have discussed the problem of defining, in a general Riemannian space-time, the concept of distance between two particles, as distinct from that of interval (or integrated line-element) between two events. The problem has also been considered by Dr. R. C. Tolman [§] with reference to particular metrics. Ruse's procedure is purely mathematical, being a natural extension of the concept of spatial distance in Special Relativity. Whittaker and Tolman, on the other hand, related their definitions to the astronomical methods of calculating great distances, such as those of the extragalactic nebulæ. These methods depend ultimately on a comparison of absolute and apparent brightness, it being assumed that brightness decreases with the square of the distance ; or, alternatively, on a similar

* Communicated by Prof. E. T. Whittaker, Sc.D., F.R.S.
† Proc. Roy. Soc. A, cxxxiii. p. 93 (1931).
‡ Proc. Roy. Soc. Edinburgh, lii. p. 183 (1932).
§ (i.) Astrophys. Journ. lxix. p. 245 (1929). (ii.) Proc. Nat. Acad. Sci. xvi. p. 511 (1930).

comparison of absolute and apparent size. It is the purpose of the present paper to investigate definitions which translate that procedure more exactly. The distance computed from brightness will be denoted by $\Delta$, and the other by $\Delta'$.

By distance we understand an invariant relating two world-points situated on definite world-lines representing two particles, which we may call respectively the " star " and the " observer." To have astronomical significance, the definition must relate two world-points on a null-geodesic ; for astronomical observations are necessarily made by rays of light, that is along null-geodesics in space-time. Thus the distance can only be measured when the interval is zero.

Unlike interval, distance is not symmetrical between the two particles ; for, in the first place, the null-geodesic must represent light from the star to the observer and not *vice versâ* ; and, secondly, their world-lines may be involved differently. In fact, it will be shown that $\Delta$ involves only the world-direction of the star, not that of the observer. On the other hand, $\Delta'$ and the distances of Whittaker and Ruse involve only the world-direction of the observer. In this way they are generalizations of the idea of spatial distance in Special Relativity ; this depends on the observer's instantaneous separation of space and time, and is measured in his instantaneous space, which is the three-dimensional manifold orthogonal to his world-line. In General Relativity this separation is only valid locally, the observer's " local space " consisting of those neighbouring world-points which he regards as simultaneous.

### § 2. *Distance computed from Estimates of Brightness.*

Whittaker's definition, intended to express distance as computed from brightness, may be expressed as follows :

We consider a thin pencil of null-geodesics (rays of light) which issue from a star at the world-point A and pass close to an observer at the world-point B. This pencil intersects the observer's local space in a two-dimensional cross-section. For observers at different points on the null-geodesic AB, the apparent brightness of the star is inversely proportional to the area of this cross-section. Accordingly the " distance AB" is defined to be proportional to the square root of the cross-section. The factor of proportionality is to be independent of the actual position of B on AB, and is to be such that when A and B are close together, " AB " shall reduce to spatial distance in the observer's local space.

My objection to Whittaker's definition is based on a criticism of this factor of proportionality, which is inconsistent with the conception of absolute brightness, and leads to a formula for distance which is not properly invariant.

We recall that absolute brightness is defined astronomically as the brightness of the star as it would appear to an observer at some small standard distance from it. Thus we really postulate a fictitious observer at a world-point $A_1$ on the same null-geodesic $AB$ (so that he observes the same pencil of light), but in the neighbourhood of the star ; and we are imagining the estimate of brightness to be made by him. If $\Theta$ and $\theta$ (see figure) are the areas of the cross-sections of the same thin pencil as observed at $B$ and $A_1$, and if $\delta$ is the distance $AA_1$ in the local space of the fictitious observer, then the distance $AB$ as computed astronomically is

$$\Delta = (\Theta/\theta)^{\frac{1}{2}}\delta, \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (1)$$

Thus the factor of proportionality is $\delta/\theta^{\frac{1}{2}}$. This depends not only on the positions of $A$ and $A_1$, but also on the unspecified world-direction associated with $A_1$, *i. e.*, on the motion of the fictitious observer. How are we to consider him to be moving?

The answer is supplied by the consideration that the factor of proportionality $\delta/\theta^{\frac{1}{2}}$, which measures the brightness at unit distance, has to be an intrinsic property of the pencil of light from the star, independent of the observer. It is therefore evident that we must give our fictitious observer the motion of the star; this is possible without ambiguity if $\delta$ is infinitesimal ; and this, with equation (1), completes our definition of distance. It is important to remark that when the star and the original observer are close together, the distance as we have now defined it reduces to spatial distance in the local space of the star, not of the observer. This follows from the fact (proved in the next section) that the apparent brightness is independent of the motion of the observer *, so that in the above case $\Theta$ reduces to $\theta$ ; the motion of the star, however, is required to determine $\delta$.

The procedure which Whittaker adopts, in applying his definition to the de Sitter world, is equivalent to assuming that the world-line of the fictitious observer is " parallel " to the world-line of the original observer in the sense of intersecting it in an arbitrarily chosen " point at infinity ; "

* This fact was pointed out to me in conversation by A. G. Walker. *Cf.* his paper about to appear in Oxford Quart. Journ. of Maths. for another treatment of the problem of this paper.

and his " distance " involves this arbitrary point on the observer's world-line instead of involving the world-line of the star. It may be shown that, by a suitable choice of the coordinates * $(u, x, y, z)$, any point on the observer's world-line

$$x = x_0, \quad y = y_0, \quad z = z_0,$$

can be taken as the point where $u$ is infinite ; and Whittaker's formula is only invariant for those transformations which preserve this point at infinity.

Tolman's procedure † may be criticized on similar grounds. Instead of comparing the apparent brightness of the star with a similar observation made by a fictitious observer in the neighbourhood of the star, he compares it with a similar observation on a fictitious star in the neighbourhood of the observer. Here, again, my criticism is that the comparison brightness is not an intrinsic property of the star, for it depends on the shape of space-time round the observer at the instant when he receives the light. Thus, in Tolman's calculations, it involves the function $y(t)$ of the cosmic time which figures in his metric evaluated at the instant of reception of the light by the observer.

## § 3. *Formula for Distance in Normal Coordinates.*

Choose a system of normal coordinates ‡ with origin at A, the world-point of the star at the instant when light is emitted : and let the metric of space-time be

$$ds^2 = g_{ij} dx^i dx^j. \quad (i, j = 0, 1, 2, 3). \quad . \quad . \quad (2)$$

(Summation with respect to repeated indices is implied.) Two properties of normal coordinates concern us here :

(i.) Geodesics through the origin are represented by linear equations.

(ii.) At any point $(x^i)$, using $\bar{g}_{ij}$ to denote the values of $g_{ij}$ at the origin,

$$g_{ij} x^j = \bar{g}_{ij} x^j. \quad . \quad . \quad . \quad . \quad . \quad (3)$$

* See § 6, and Whittaker, *loc. cit.*

† *Loc. cit.* (ii.).  His conclusion in (i.), which refers to the de Sitter world, is confirmed in § 6.

‡ An account of these is given by Veblen, ' Invariants of Quadratic Differential Forms,' Camb. Math. Tracts, No. 24 (1927), chap. 6.

Let the null-geodesic AB be

$$x^i = \xi^i \lambda, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $\xi^i$ are four constants satisfying the null relation

$$\bar{g}_{ij}\xi^i\xi^j = 0, \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which, by (3), can also be written

$$g_{ij}\xi^i\xi^j = 0. \quad . \quad . \quad . \quad . \quad . \quad (6)$$

For the neighbouring null-geodesic AB′, with the equation

$$x^i = (\xi^i + d\xi^i)\lambda, \quad . \quad . \quad . \quad . \quad . \quad (7)$$

we have, by differentiating (5),

$$\xi_i d\xi^i = 0, \quad . \quad . \quad . \quad . \quad . \quad . \quad (8)$$
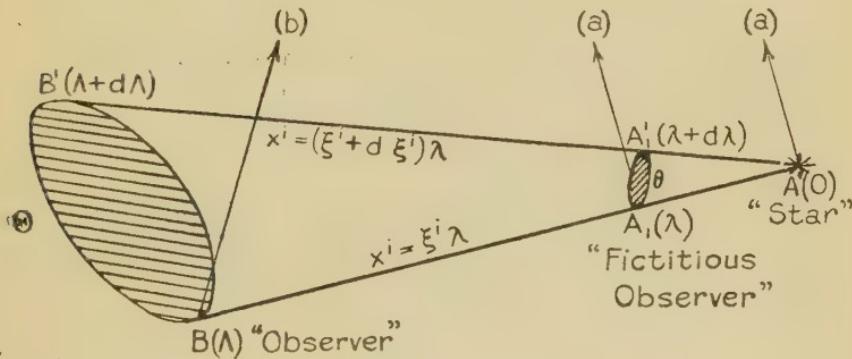


where

$$\xi_i = \bar{g}_{ij}\xi^j = g_{ij}\xi^j. \quad . \quad . \quad . \quad . \quad (9)$$

Now if **b** is the vector specifying the observer's motion, *i. e.*, a unit vector at B directed along his world-line, then the observer's local space is

$$b_i dx^i = 0 ; \quad . \quad . \quad . \quad . \quad . \quad (10)$$

that is, a displacement $dx^i$, to be recognized by him as spatial, must satisfy (10).

Let B be the point $(x^i)$ of parameter $\Lambda$ on (4) ; and let B′, the intersection of (7) and (10), be the point $(x^i + dx^i)$ of parameter $\Lambda + d\Lambda$ on (7). Then, differentiating (4),

$$dx^i = \Lambda d\xi^i + \xi^i d\Lambda. \quad . \quad . \quad . \quad . \quad (11)$$

Substituting (11) in (10), and putting $b_i \xi^i = 1/k$, we have

$$d\Lambda = -k\Lambda b_i d\xi^i.$$

Substituting back in (11),

$$dx^i = \Lambda d\xi^i - k\Lambda \xi^i b_h d\xi^h.$$

Substituting in (2),

$$-BB'^2 = \Lambda^2 g_{ij} d\xi^i d\xi^j, \quad . \quad . \quad . \quad . \quad (12)$$

the terms involving the first power of $k$ vanishing by (8), and those involving $k^2$ by (6).

So far we have only specified the ratios of the constants $\xi^i$. To fix their values, let

$$c_i \xi^i = 1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

where $c_i$ are four constants; and let this remain true in the variation, so that

$$c_i d\xi^i = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

Take coordinates $\eta_1$, $\eta_2$ in the wave-front (the locus of B'), these being linear functions of the $\xi$'s, so that

$$d\eta_\alpha = c_{\alpha i} d\xi^i, \quad (\alpha, \beta = 1, 2), \quad . \quad . \quad . \quad (15)$$

where $c_{\alpha i}$ are eight constants.

Let $\Xi$ denote the determinant of the sixteen quantities $\xi_i$, $c_i$, $c_{\alpha i}$; and assume that $c_i$, $c_{\alpha i}$ have been chosen so that $\Xi \neq 0$. Then we can solve the four equations (8), (14), (15) for $d\xi^i$, obtaining

$$d\xi^i = c^{\alpha i} d\eta_\alpha,$$

where $c^{\alpha i} = (\text{minor of } c_{\alpha i} \text{ in } \Xi) \div \Xi$.

Substituting in (12), we get for the linear element in the wave-front:

$$-BB'^2 = \gamma^{\alpha\beta} d\eta_\alpha d\eta_\beta, \quad (\alpha, \beta = 1, 2),$$

where

$$\gamma^{\alpha\beta} = \Lambda^2 g_{ij} c^{\alpha i} c^{\beta j}.$$

Hence the element of area is

$$\begin{aligned}
\Theta &= (\gamma^{11}\gamma^{22} - \gamma^{12}\gamma^{12}) d\eta_1 d\eta_2 \\
&= \Lambda^2 (g_{ij} c^{1i} c^{1j} \cdot g_{kl} c^{2k} c^{2l} - g_{ij} c^{1i} c^{2j} \cdot g_{kl} c^{1k} c^{2l})^{\frac{1}{2}} d\eta_1 d\eta_2 \\
&= \Lambda^2 \Phi^{\frac{1}{2}} d\eta_1 d\eta_2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)
\end{aligned}$$

where

$$\Phi = g_{ij} g_{kl} c^{1i} c^{2l} (c^{1j} c^{2k} - c^{1k} c^{2j}).$$

From the anti-symmetry of this expression in $j$ and $k$, it can be written

$$\Phi = \tfrac{1}{2}(g_{jl}g_{hl} - g_{ik}g_{ij})c^{1i}c^{2l}(c^{1j}c^{2k} - c^{1k}c^{2j}),$$

and this again, from the anti-symmetry in $i$ and $l$, can be written

$$\Phi = \tfrac{1}{4}(g_{ij}g_{kl} - g_{ik}g_{jl})(c^{1i}c^{2l} - c^{1}c^{2i})(c^{1j}c^{2k} - c^{1k}c^{2j}),$$

Now any second-order minor of a determinant is equal to the cofactor of the corresponding minor of the adjugate determinant, divided by the original determinant. Thus

$q_{ij}g_{hl} - g_{ik}g_{jl} = g \times$ the cofactor of $(g^{ij}g^{kl} - g^{ik}g^{jl})$ in $|g^{ij}|$,

$c^{1i}c^{2j} - c^{1j}c^{2i} = \Xi^{-1} \times$ the cofactor of $(c_{1i}c_{2j} - c_{1i}c_{2i})$ in $\Xi$.

It follows that

$$\begin{aligned}
\Phi &= \tfrac{1}{4}g\Xi^{-2}(g^{ij}g^{kl} - g^{ik}g^{jl})(c_i\xi_l - c_l\xi_i)(c_j\xi_k - c_k\xi_j)\\
&= \tfrac{1}{2}g\Xi^{-2}(g^{ij}g^{kl} - g^{ik}g^{jl})c_i\xi_l(c_j\xi_k - c_k\xi_j)\\
&= g\Xi^{-2}g^{ij}g^{kl}c_i\xi_l(c_j\xi_k - c_k\xi_j)\\
&= g\Xi^{-2}g^{ij}g^{kl}c_ic_j\xi_l\xi_k - g\Xi^{-2}g^{ij}g^{kl}c_ic_k\xi_j\xi_l.
\end{aligned}$$

The first term vanishes by (6) and (9). Hence

$$\begin{aligned}
\Phi &= -g\Xi^{-2}(c_i\xi^i)^2\\
&= -g\Xi^{-2}, \text{ by (13).}
\end{aligned}$$

Substituting in (16),

$$\Theta = \Lambda^2(-g)^{\tfrac{1}{2}}\Xi^{-1}d\eta_1 d\eta_2. \quad \cdots \quad (17)$$

It will be observed that the motion **b** of the observer does not enter into this expression, and consequently is not involved either in the apparent brightness of the star or in the distance.

Now consider the " fictitious observer " in the immediate neighbourhood of the star. Let him be at the point $A_1$ of parameter $\lambda$ on (4). His spatial distance from the star, measured in the local space of the star, is

$$\delta = a_i\lambda\xi^i,$$

where **a** is the vector specifying the motion of the star. For this represents the projection of $AA_1$ on the world-line of the star; and since $A$ and $A_1$ are on the same null-

geodesic, their space-separation is equal to their time-separation *.

Analogously to (17),

$$\theta = \lambda^2(-\bar{g})^{\frac{1}{2}}\Xi^{-1}d\eta_1 d\eta_2.$$

The factor $\Xi^{-1}d\eta_1 d\eta_2$ depends on the pencil and not on the position of B ; thus it cancels on substitution in (1), and we obtain finally

$$\Delta = a_i x^i (g/\bar{g})^{\frac{1}{4}}. \quad . \quad . \quad . \quad . \quad . \quad (18)$$

This gives the distance (computed from brightness) of a star at the origin from an observer at $(x^i)$ in normal coordinates ; $a_i$ is the unit covariant vector in the direction of the star's world-line ; and $g, \bar{g}$ are the values of the determinant $\|g_{ij}\|$ at the world-points of the observer and the star respectively.

## § 4. *Generalization to an Arbitrary System of Coordinates.*

Let $\Omega$ denote half the square of the interval (along a geodesic) between the two world-points $(x^i)$ and $(\bar{x}^i)$ in any coordinate system. Regarded as a function either of the $x$'s or of the $\bar{x}$'s, $\Omega$ is an absolute scalar, having the same value in any coordinate system. Taking $(x)$ and $(\bar{x}^i)$ to be the world-points of the observer and the star respectively, $\Omega$ vanishes : but its derivatives with respect to $x^i$ and $\bar{x}^i$ do not necessarily vanish, and we can form the determinant †

$$\mathbf{J} = \|\partial^2\Omega/\partial\bar{x}^i\partial x^j\|.$$

The equations of transformation of J, $g, \bar{g}$, are :

$$\mathbf{J}' = \left\|\frac{\partial\bar{x}^i}{\partial\bar{x}'^k} \frac{\partial x^j}{\partial x'^l} \frac{\partial^2\Omega}{\partial\bar{x}^i\partial x^j}\right\|$$

$$= \frac{\partial(\bar{x})}{\partial(\bar{x}')} \frac{\partial(x)}{\partial(x')} \mathbf{J} \; ;$$

$$g' = \frac{\partial(x)}{\partial(x')} g \; ;$$

$$\bar{g}' = \frac{\partial(\bar{x})}{\partial(\bar{x}')} \bar{g}.$$

---

* It is assumed that the coordinates are such that $c$, the velocity of light, is unity. Otherwise the expressions for $\delta$ and $\triangle$ have to be multiplied by $c$.

† *Cf.* Ruse, Proc. Edinburgh Math. Soc. ii. p. 135 (1931).

It follows that the expression $J^2/g\bar{g}$ is an absolute scalar like $\Omega$. Again, regarding $(x^i)$ as fixed and $(\bar{x}^i)$ as a variable point on the world-line of the star, $\Omega$ is a function of the proper time $\bar{\tau}$ of the star; and the function $\partial\Omega/\partial\bar{\tau}$ is another absolute scalar. It will now be shown that $\Delta$ can be expressed in terms of $J^2/g\bar{g}$ and $\partial\Omega/\partial\bar{\tau}$.

In normal coordinates with origin $(\bar{x}^i)$, we have *

$$\frac{\partial\Omega}{\partial\bar{x}} = -\bar{g}_{ij}x^j.$$

Thus

$$\frac{\partial^2\Omega}{\partial\bar{x}^i\partial x^j} = -\bar{g}_{ij},$$

whence

$$J = \bar{g},$$

and

$$\frac{J^2}{g\bar{g}} = \frac{\bar{g}}{g}. \quad \ldots \ldots \quad (19)$$

Also

$$\frac{\partial\Omega}{\partial\bar{\tau}} = a^j\frac{\partial\Omega}{\partial\bar{x}^j} = -\bar{g}_{ij}x^ia^j = -a_ix^i.$$

Comparing (18), we have

$$\Delta = -\frac{\partial\Omega}{\partial\bar{\tau}}\left(\frac{g\bar{g}}{J^2}\right)^{\frac{1}{4}}. \quad \ldots \ldots \quad (20)$$

We have proved this for a normal coordinate system, but the preceding considerations show that it is true generally.

Introducing the proper time $\tau$ of the observer, the Doppler red-shift in the stellar spectrum is given by

$$1 + \frac{\delta\lambda}{\lambda} = \frac{d\bar{\tau}}{d\tau} = -\frac{\partial\Omega}{\partial\bar{\tau}}\Big/\frac{\partial\Omega}{\partial\tau},$$

where $\lambda$, $\lambda + \delta\lambda$ are the actual and observed wave-lengths. Thus (20) can be written in the alternative form

$$\Delta = \left(1 + \frac{\delta\lambda}{\lambda}\right)\frac{\partial\Omega}{\partial\tau}\left(\frac{g\bar{g}}{J^2}\right)^{\frac{1}{4}} \quad \ldots \ldots \quad (21)$$

In applications of these formulæ, use can be made of the equation $\Omega = 0$ to eliminate one of the variables or otherwise to simplify the result.

It may be mentioned that the function $\partial\Omega/\partial\tau$ is the distance AB as defined by Ruse †. For practical purposes

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* Ruse, Proc. London Math. Soc. xxxii. p. 87 (1931), equation (12).
† *Loc. cit.* § 1.

our formula probably does not differ greatly from Ruse's.
For, in the first place, neglect of the factor $(1 + \delta\lambda/\lambda)$ would
not make much difference to present estimates of nebular
distances, the greatest Doppler effects so far observed being
of the order $\delta\lambda/\lambda = 1/10$. In the second place, there is reason
to believe that space-time has approximately the form of
the de Sitter world, in which $J^2/g\bar{g}$ is unity when $\Omega$ is zero.
(See § 6.)

## § 5. *Distance computed from Estimates of Size.*

For the distance $\Delta'$, computed from estimates of size, we
have to consider rays of light issuing from different points
of the star, and reaching the observer simultaneously, *i. e.*, a
thin pencil of null-geodesics passing near A and converging
at B. If the apparent dimensions are measured by the solid
angle subtended at B, *i. e.*, by the area intersected on a small
sphere round B of radius $\delta$, and if this is compared with the
actual area under observation as measured by someone
moving with the star, then it will be found that our original
definition is exactly reversed, and the formula for this
distance $\Delta'$ will be given by interchanging the rôles of star
and observer in the formula for $\Delta$, with this difference, that
in both cases the null-geodesic must be such as to represent
light from the star to the observer. Thus from (20) and
(21), we deduce

$$\Delta' = \frac{\partial\Omega}{\partial\tau}\left(\frac{g\bar{g}}{J^2}\right)^{\frac{1}{4}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

$$= -\left(1 + \frac{\delta\lambda}{\lambda}\right)^{-1}\frac{\partial\Omega}{\partial\bar{\tau}}\left(\frac{g\bar{g}}{J^2}\right)^{\frac{1}{4}},$$

whence

$$\Delta = \left(1 + \frac{\delta\lambda}{\lambda}\right)\Delta'. \quad . \quad . \quad . \quad . \quad . \quad (23)$$

Two conclusions follow from the last equation :—

(i.) The distance computed from estimates of brightness
will agree exactly with the distance computed from estimates
of size only when the star has no apparent radial velocity.

(ii.) For spherical objects of the same absolute size and
brightness, moving variously at varying distances from the
observer, the ratio of the apparent diameter to the square
root of the apparent brightness is proportional to $(1 + \delta\lambda/\lambda)$,

instead of being constant as is ordinarily assumed. This might be made the subject of an observational test for either stars or nebulæ. For instance, Hubble * arranged the extragalactic nebulæ in a series of groups; in each group the absolute size and brightness are found to be nearly constant; perhaps, therefore, a correlation could be found between the above ratio and the expression $(1 + \delta\lambda/\lambda)$. The test could be applied to the first group of nebulæ in the series, as they are approximately spherical (so that the effects of random orientation need not be considered); and would provide, if successful, a confirmation of the theory of Relativity. A similar test was proposed by Tolman †, with, however, two differences:

(a) He found the ratio proportional to $(1 + \delta\lambda/\lambda)^2$, the discrepancy being due to his different interpretation of "absolute brightness," referred to in § 2.

(b) He proposed it as a test of the particular type of Expanding Universe which he was considering, although it would apply equally to any Riemannian space-time.

## § 6. *Application to the de Sitter World.*

Following Whittaker, we take the metric of the de Sitter world in the form

$$\frac{ds^2}{R^2} = \frac{du^2 - dx^2 - dy^2 - dz^2}{1 + x^2 + y^2 + z^2 - u^2} + \frac{(u\,du - x\,dx - y\,dy - z\,dz)^2}{(1 + x^2 + y^2 + z^2 - u^2)^2}, \quad (24)$$

for which ‡

$$\cosh\frac{\sqrt{2\Omega}}{R} = \frac{1 + x\bar{x} + y\bar{y} + z\bar{z} - u\bar{u}}{(1 + x^2 + y^2 + z^2 - u^2)^{\frac{1}{2}}(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)^{\frac{1}{2}}};$$

whence

$$1 + \frac{2\Omega}{R^2} + \frac{4\Omega^2}{3R^4} + \cdots$$

$$= \frac{(1 + x\bar{x} + y\bar{y} + z\bar{z} - u\bar{u})^2}{(1 + x^2 + y^2 + z^2 - u^2)(1 + \bar{x}^2 + \bar{y}^2 + \bar{z}^2 - \bar{u}^2)}. \quad (25)$$

* Astrophys. Journ. lxiv. p. 321 (1926).
† *Loc. cit.* (ii.).
‡ Whittaker, *loc. cit.* (equation (4)).

The origin of these coordinates is arbitrary. Taking the star to be at the origin, so that

$$\bar{x} = \bar{y} = \bar{z} = \bar{u} = 0, \quad \ldots \quad (26)$$

the condition $\Omega = 0$ gives

$$x^2 + y^2 + z^2 = u^2. \quad \ldots \quad (27)$$

Performing the necessary partial differentiations on (25), and then substituting (26) and (27), we find:

$$J = \left\| \frac{\partial^2 \Omega}{\partial \bar{x}_i \partial x_j} \right\| = R^8 \begin{vmatrix} 1 - 2x^2, & -2xy, & -2zx, & 2ux \\ -2xy, & 1 - 2y^2, & -2yz, & 2uy \\ -2zx, & -2yz, & 1 - 2z^2, & 2uz \\ 2ux, & 2uy, & 2uz, & -1 - 2u^2 \end{vmatrix}$$

$$= R^8(-1 + 2x^2 + 2y^2 + 2z^2 - 2u^2)$$

$$= -R^8.$$

We have also

$$g = R^8 \begin{vmatrix} -1 + x^2, & 2xy, & 2zx, & -2ux \\ 2xy, & -1 + y^2, & 2yz, & -2uy \\ 2zx, & 2yz, & -1 + z^2, & -2uz \\ -2ux, & -2uy, & -2uz, & 1 + u^2 \end{vmatrix}$$

$$= R^8(-1 + x^2 + y^2 + z^2 - u^2)$$

$$= -R^8;$$

$$\bar{g} = R^8 \begin{vmatrix} -1, & 0, & 0, & 0 \\ 0, & -1, & 0, & 0 \\ 0, & 0, & -1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} = -R^8.$$

Thus

$$\frac{J^2}{gg} = 1,$$

and this must be true generally in the de Sitter world for any two points on a null-geodesic.

It now follows from (20), (21), and (22) that

$$\Delta' = \frac{\partial \Omega}{\partial \bar{\tau}},$$

$$\Delta = \left( 1 + \frac{\delta \lambda}{\lambda} \right) \frac{\partial \Omega}{\partial \tau} = -\frac{\partial \Omega}{\partial \bar{\tau}}.$$

Thus, in the de Sitter world,

(i.) $\Delta'$ coincides with Ruse's distance ;

(ii.) $\Delta$ differs from it by the factor $(1+\delta\lambda/\lambda)$ ;

(iii.) the distance $\Delta$ of a star from an observer coincides with Ruse's distance of the observer from the star.

Applying the transformation

$$x = r(\mathrm{R}^2 - r^2)^{-\frac{1}{2}} \sin\theta \cos\phi \operatorname{sech}(t/\mathrm{R}),$$
$$y = r(\mathrm{R}^2 - r^2)^{-\frac{1}{2}} \sin\theta \sin\phi \operatorname{sech}(t/\mathrm{R}),$$
$$z = r(\mathrm{R}^2 - r^2)^{-\frac{1}{2}} \cos\theta \operatorname{sech}(t/\mathrm{R}),$$
$$u = \tanh(t/\mathrm{R}),$$

we obtain from (24) the metric of the de Sitter world in the more usual form

$$ds^2 = (1 - r^2/\mathrm{R}^2)dt^2 - (1 - r^2/\mathrm{R}^2)^{-1}dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2.$$

Let $r$ be the coordinate of the star when the observer is taken to be at rest at the spatial origin ; and let $r_0$ be the coordinate of the observer when, on a transformation of coordinates, the star is taken to be at rest at the spatial origin. Then it has already been shown by Ruse [*] that

$$\frac{\partial\Omega}{\partial\tau} = r.$$

Hence we have the following results for the distances $\Delta$, $\Delta'$ in the de Sitter world :

$$\Delta = r_0 = \left(1 + \frac{\delta\lambda}{\lambda}\right)r ;$$

$$\Delta' = r = \left(1 + \frac{\delta\lambda}{\lambda}\right)^{-1} r_0.$$

These results agree with Tolman's conclusions [†].

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LXI. *On an Instrument for converting Voltage Values into* "$H\rho$" *Values for Electrons, and* vice versa, *according to Relativity Formulæ.* By J. V. Hughes, *A.R.C.S., B.Sc.*[‡]

IF $e_m$, $e_s$, be the charge on the electron in e.m.u. and e.s.u. respectively, $m$, $m_0$, the mass of the electron when travelling with a velocity $v$ and when at rest respectively,

[*] *Loc. cit.* § 1.

[†] *Loc. cit.* § 1 (i.).

[‡] Communicated by Prof. G. P. Thomson, M.A.

V the accelerating potential which gave rise to the velocity in the electron, and $\rho$ the radius of curvature of the path traced out by the electron when moving perpendicular to a magnetic field of strength H ; then we have

$$mv^2/\rho = He_m v,$$

or
$$H\rho = mv/e_m = m_0 c\beta/e_m\sqrt{1-\beta^2}, \quad . \quad . \quad . \quad (1)$$

where $\beta = v/c$ and $c$ is the velocity of light.

Also

$$Ve_s = 300(mc^2 - m_0 c^2) = 300 m_0 c^2(1/\sqrt{1-\beta^2} - 1), . \quad (2)$$

the 300 being introduced since V is in volts and the rest of the equation refers to e.s.u.

(1) can be written

$$(1-\beta^2)(H\rho)^2 = (m_0 c/e_m)^2\beta^2,$$

or
$$\beta^2\{(H\rho)^2 + (m_0 c/e_m)^2\} = (H\rho)^2.$$

Thus

$$\beta = H\rho/\sqrt{(H\rho)^2 + (m_0 c/e_m)^2}. \quad . \quad . \quad . \quad (3)$$

From (3), we get

$$1 - \beta^2 = 1 - (H\rho)^2/\{(H\rho^2) + (m_0 c/e_m)^2\}$$
$$= (m_0 c/e_m)^2/\{(H\rho)^2 + (m_0 c/e_m)^2\},$$

or

$$1/\sqrt{1-\beta^2} = \sqrt{(H\rho)^2 + (m_0 c/e_m)^2}/(m_0 c/e_m).$$

Substituting in (2), we obtain

$$Ve_s = 300 m_0 c^2\{\sqrt{(H\rho)^2 + (m_0 c/e_m)^2}/(m_0 c/e_m) - 1\},$$

or

$$Ve_s = 300 c e_m\{\sqrt{(H\rho)^2 + (m_0 c/e_m)^2} - (m_0 c/e_m)\}.$$

But $e_m/e_s = 1/c$. Hence

$$V = 300\{\sqrt{(H\rho)^2 + (m_0 c/e_m)^2} - (m_0 c/e_m)\},$$

which is our working equation.

Consider a triangle right-angled at B, and with $BC = m_0 c/e_m$ to some convenient scale. Let AB represent an $H\rho$ value to the same scale.

Then

$$CA = \sqrt{(H\rho)^2 + (m_0 c/e_m)^2},$$

so that if $DC = CB = m_0 c/e_m$, we see that $AD = V/300$, to the same scale that AB represents $H\rho$ and BC represents $m_0 c/e_m$.

This is true for all values of H$\rho$. Thus if a scale of H$\rho$ be set up along BA (on the same scale that BC = $m_0c/e_m$) and an arm be pivoted at C with a scale, commencing at D (where DC = CB) marked along it, also on the same scale, then the readings on the two scales at the point of intersection give corresponding values of H$\rho$ and V/300 respectively.

Furthermore, we see from equation (3) that

$$\beta = H\rho / \sqrt{(H\rho)^2 + (m_0c/e_m)^2} = AB/AC = \sin A\hat{C}B,$$

Thus the ordinate of a fixed point on AC gives $\beta$ on the scale for which the distance of the fixed point from C is unity.
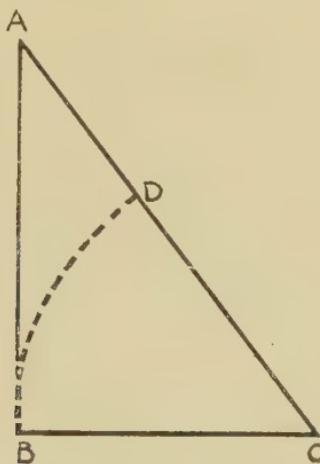


The instrument brings out very clearly two points :—

(*a*) At small values of the energy, a small change in the energy gives rise to a large change in value of H$\rho$ and also in that of $\beta$. At large energy values the changes of energy and of H$\rho$ are more proportionate, and $\beta$ changes comparatively slowly.

(*b*) The only effect of a change in the value of $e_m/m_0$ is to change the distance of the pivot C from B ; at large energy values it is readily seen that this effect is small.

A rough preliminary model has given satisfactory results. Plans are now in hand for a more accurate model, fitted with verniers on both scales, reading to 0·2 unit of H$\rho$, and 0·1 KV. The position of the pivot relative to both scales is to be adjustable. An advantage of the instrument is that all scales are evenly divided, which carries with it also the possibility of the use of verniers.

LXII. *Deflexions and Vibrations of a Circular Elastic Plate under Tension.* By W. G. BICKLEY, *D.Sc., Imperial College of Science and Technology* *.

## *Introduction.*

THE problem to be studied in this paper arose out of the desire to construct a form of condenser microphone to be used for measurement, in which the resonant frequency of the diaphragm should be well above at least the more important audio frequencies, so that response should be practically independent of frequency. To do this it was proposed to subject the diaphragm to considerable tension in its plane, and both for design purposes, and also later in use (if successful), the effects of this tension upon the deflexion of the diaphragm under pressure, and upon the fundamental natural frequency, were required. The present paper presents solutions of these and related problems ; but the application of the mathematical results to the particular technical problem mentioned above is left to others.

Attention has been concentrated upon a circular plate clamped at the edge, since this shape and these boundary conditions are of the most practical importance. Although a " supported " edge is mathematically conceivable, it is not so easy to conceive any form of support which could apply a tension without in some way constraining the angular deviation at the edge, and thus applying some bending moment there. If, for instance, the flat end of a cylindrical drum under internal pressure is considered, tension being applied by the cylindrical portion, the conditions at the junction would be, in practice, intermediate between those appropriate to " clamped " and " supported " edges. It is possible that this problem may be solved later.

## 1. *Fundamental Equations.*

The deflexion and free vibrations of an elastic plate, based upon the " approximate theory," *i. e.*, that in which only average or resultant stresses across a section are taken into account, are well known, and the results can be found in many books †. If, in addition to being elastic, the plate is

* Communicated by the Author.

† See, for example, Love, ' Mathematical Theory of Elasticity ' ; Rayleigh, ' Sound,' i. ; Prescott, ' Applied Elasticity '; Case, ' Strength of Materials.'

in tension, it is clear that the deflexion under any applied load will be reduced, while the frequency of any normal mode of vibration will be raised. If the tension is predominant, we may expect the equations to approximate to those of a membrane, and these equations and their solutions are again well known [*]. For the intermediate cases whose consideration is the subject of this paper, the fundamental equations can be found—

   (*a*) by a synthesis of the known equations embodying the two types of restoring force, whose effects, for small displacements (*small* displacements will be tacitly assumed throughout), are additive ;

   (*b*) by a variational method from the expression of the total potential energy due to elasticity and tension, and kinetic energy ;

   (*c*) from first principles, by considering the forces and couples acting on an element of the plate.

It seems hardly worth while to traverse ground which is already well surveyed, so we content ourselves here by giving the results which would be obtained by either (*a*), (*b*), or (*c*) above, without their detailed deduction.

We denote by :

   $2h$,   the thickness of the plate, assumed uniform ;

   $E, \sigma$,   Young's Modulus and Poisson's Ratio, respectively, of the material of which it is made ;

   $\rho$,   the density of this material ;

   $x, y$,   rectangular coordinates in the plane of the plate ;

   $r, \theta$,   polar ditto ;

   $ds, dn$,   elements of the arc, and outward drawn normal, of a curve in the plane of the plate ;

   $t$,   time ;

   $w$,   the normal displacement of the middle surface at any point ;

$B = 2Eh^3/3(1-\sigma^2)$, the appropriate flexural rigidity ;

   $T$,   the tension ;

   $p$,   pressure, or normal force per unit area.

---

[*] Rayleigh, 'Theory of Sound,' ch. ix.; Lamb, 'Dynamical Theory of Sound,' ch. v.

The fundamental differential equation which is satisfied over the surface of the plate is

$$-B\nabla^4 w + T\nabla^2 w + p = 2ph\frac{\partial^2 w}{\partial t^2}, \quad . \quad . \quad (1.1)$$

where $\nabla^2$ denotes the two-dimensional form of Laplace's operator, *i. e.*,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \quad . \quad (1.2)$$

If we consider a portion of the plate bounded by a curve, the resultant of the tractions exerted upon this portion by the remainder, over an element $\delta s$ of the boundary, will have components—

(*a*) a force $N_s\delta s$ in the direction of *w*, with

$$N_s = \frac{\partial}{\partial n}\{Tw - B\nabla^2 w\}; \quad . \quad . \quad . \quad (1.31)$$

(*b*) a bending moment, $G_s\delta s$, which will be regarded as positive with the usual convention of sense if *dn*, *ds*, and *w* form a right-handed set of axes, with

$$G_s = -B\left\{\frac{\partial^2 w}{\partial n^2} + \sigma\left(\frac{1}{\rho}\frac{\partial w}{\partial n} + \frac{\partial^2 w}{\partial s^2}\right)\right\}. \quad . \quad (1.32)$$

$\rho$ being here the radius of curvature of the arc of the boundary ;

(*c*) a torque, $H_s\delta s$, about the normal, with

$$H_s = B(1-\sigma)\frac{\partial}{\partial n}\left(\frac{\partial w}{\partial s}\right). \quad . \quad . \quad (1.33)$$

### 2. *Deflexion of a Taut Circular Plate clamped at the Edge, subjected to a Uniform Pressure.*

For the solution of this problem we need a solution of the equation

$$B\nabla^4 w - T\nabla^2 w = p, \quad . \quad . \quad . \quad (2.1)$$

subject to the conditions that *w* and $\partial w/\partial r$ vanish when $r=a$, the radius of the plate. By symmetry, *w* is independent of $\theta$. A particular integral of the equation is

$$w_1 = -\frac{p}{2T}r^2. \quad . \quad . \quad . \quad . \quad (2.2)$$

If we write $k^2 = T/B$, the differential equation for the complementary function may be written

$$(\nabla^2 - k^2) \cdot \nabla^2 w = 0, \quad \ldots \quad (2.31)$$

so that the complementary function is

$$w_2 = A \log r + C + D I_0(kr) + F K_0(kr), \quad (2.32)$$

where $I_0$ and $K_0$ denote Bessel functions of imaginary argument, of the first and second kinds respectively, the notation being that of Watson's 'Bessel Functions.' The complete value of $w$ is $w_1 + w_2$.

The condition that the deflexion is finite at the origin necessitates that $A = F$. The condition that the total normal force round a circle of radius $r$, i. e.,

$$2\pi r N_r = -2\pi r B \frac{\partial}{\partial r} (\nabla^2 - k^2) w,$$

tends to zero with $r$, necessitates $A = 0$. The fulfilment of the conditions at the boundary $r = a$ leads to the equations

$$\left. \begin{array}{l} -\dfrac{pa^2}{4T} + C + D I_0(ka) = 0, \\[2mm] -\dfrac{pa}{2T} + k D I_0{}'(ka) = 0, \end{array} \right\} \quad \ldots \quad (2.41)$$

so that

$$D = \frac{pa}{2k T I_0{}'(ka)} = \frac{pa^2}{2T} \cdot \frac{1}{ka I_0{}'(ka)}, \quad \ldots \quad (2.42)$$

$$C = \frac{pa^2}{4T} \left\{ 1 - \frac{2 I_0(ka)}{ka I_0{}'(ka)} \right\} \quad \ldots \quad \ldots \quad (2.43)$$

Using the relation

$$I_0{}'(z) = I_1(z), \quad \ldots \quad \ldots \quad \ldots \quad (2.5)$$

the deflexion formula can be written

$$w = \frac{pa^2}{4T} \left\{ 1 - \frac{r^2}{a^2} - 2 \frac{I_0(ka) - I_0(kr)}{ka I_1(ka)} \right\}. \quad \ldots \quad (2.6)$$

When T is small (i. e., $ka$ is small) expansion in series yields the approximation,

$$w = \frac{p}{64B} (a^2 - r^2)^2 \left\{ 1 - \frac{T}{72B} (5a^2 - 2r^2) \ldots \right\}, \quad (2.7)$$

of which the first term is the well-known formula for the deflexion in the absence of tension.

When, on the other hand, $ka$ is large, the asymptotic formula

$$I_0(z) \sim (2\pi z)^{-\frac{1}{2}} e^z$$

shows that the second term of (2.6) is practically constant until $r$ approximates to $a$. The first term,

$$\frac{p}{4\mathrm{T}} (a^2 - r^2),$$

is the deflexion formula for a membrane supported at the edge and subjected to a uniform pressure $p$. Thus, as was to be expected, the central portions of the plate behave as a membrane, for all practical purposes, when the tension is great, the effects of stiffness being negligible except in a narrow annulus round the rim.

Fig. 1.
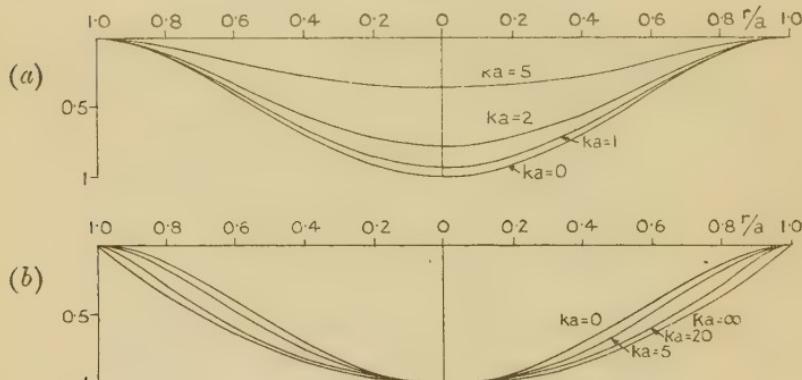


Deflexion of plate under uniform pressure.

In fig. 1 (*a*) are plotted the shapes of the meridian sections of the deflected plate, for various values of T indicated by the values of $ka$ attached to each. The way in which the deflected form approaches that of a membrane as $ka$ (or T) increases is shown in fig. 1 (*b*), where the central deflexion is taken as the unit in each case, and the meridian section of a membrane is added.

To see what these deflexions mean in practice, we take the case of a steel diaphragm of radius 2 cm. and thickness 0·02 cm. Taking $\rho = 7 \cdot 7$ gm./c.c., E as $2 \times 10^{12}$ dynes/sq. cm., and $\sigma^2$ as $0 \cdot 08$, we find

$$\mathrm{B} = 1 \cdot 45 \times 10^6 \text{ dynes/cm.,}$$

and thus

$$\mathrm{T} = (ka)^2 \, \mathrm{B}/a^2 = 3 \cdot 6 \times 10^5 \, (ka)^2 \text{ dynes/cm.}$$

With $ka = 20$, $T = 1\cdot45 \times 10^8$ dynes/cm., and the stress in the diaphragm due to T is $T/2h = 7\cdot25 \times 10^9$ dynes/sq. cm. (nearly .5 tons/sq. in.).

With a pressure of $10^6$ dynes/sq. cm. (about one atmosphere) the central deflexion in the absence of tension is

$$pa^4/64B = \cdot173 \text{ cm.,}$$

which, when $ka = 20$, is reduced to $0\cdot0062$ cm. We shall see later that it needs a value of $(ka)$ in the neighbourhood of 40 to raise the fundamental frequency of such a diaphragm to above $10{,}000 \sim$/sec., and this involves decreasing the response to pressure variations to about one-hundredth of what it would be in the absence of tension. For our disk, this would involve a stress of nearly 20 tons/sq. in. For any given value of $ka$, however, the stress varies as $h^2/a^2$, so that for a disk of greater radius or smaller thickness, the desired raising of the fundamental mode can be secured with smaller stresses—but, of course, always at the expense of sensitiveness to pressure variations. If the central portion is used to measure pressure variations, the sensitivity is inversely proportional to the tension, provided that this is appreciable.

### 3. *Concentrated Load at the Centre.*

If we have a concentrated load, P, at the centre of the diaphragm, the deflexion is given by

$$w = A\{\log r + K_0(kr)\} + C + DI_0(kr), \quad . \quad . \quad (3.1)$$

where we have already used the condition that the central deflexion is not infinite. The other condition at the centre is

$$\operatorname*{L^t}_{r \to 0} 2\pi r N_r = -P, \quad . \quad . \quad . \quad . \quad . \quad (3.21)$$

*i. e.,*

$$\operatorname*{L^t}_{r \to 0} r \frac{d}{dr}(\nabla^2 - k^2)w = \frac{P}{2\pi B} \quad . \quad . \quad . \quad (3.22)$$

or

$$-k^2 A = \frac{P}{2\pi B},$$

so that

$$A = -\frac{P}{2\pi T}. \quad . \quad . \quad . \quad (3.23)$$

The conditions at the boundary, $r = a$, are

$$w = 0 \quad \text{or} \quad -\frac{P}{2\pi T}\log a + K_0(ka) + C + DI_0(ka) = 0 ;$$

$$. \quad . \quad . \quad (3.24)$$

$$\frac{dw}{dr} = 0 \quad \text{or} \quad -\frac{P}{2\pi T} a^{-1} + k K_0'(ka) + Dk I_0'(ka) = 0 ;$$

$$. \quad . \quad . \quad (3.25)$$

whence
$$D = \frac{P}{2\pi T} \cdot \frac{1 + ka K_0'(ka)}{ka I_0'(ka)}, \quad . \quad . \quad . \quad (3.26)$$

$$C = \frac{P}{2\pi T} \frac{ka I_0'(ka)\{\log a + K_0(ka)\} - I_0(ka)\{1 + ka K_0'(ka)\}}{ka I_0'(ka)} .$$

$$. \quad . \quad . \quad (3.27)$$

Using the formulæ

$$I_0'(z) = I_1(z),$$
$$K_0'(z) = -K_1(z),$$
$$K_0(z) I_0'(z) - K_0'(z) I_0(z) = z^{-1},$$
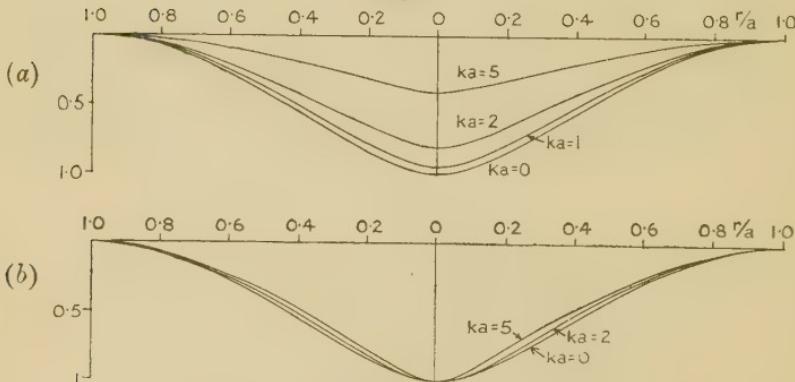
Fig. 2.



Deflexion of plate under central load.

the value of C reduces to

$$\frac{P}{2\pi T} \left\{ \log a - \frac{I_0(ka) - 1}{ka I_1(ka)} \right\} . \quad . \quad . \quad (3.28)$$

Finally,

$$w = \frac{P}{2\pi T} \left\{ \log \frac{a}{r} - K_0(kr) + \frac{1 - ka K_1(ka)}{ka I_1(ka)} I_0(kr) - \frac{I_0(ka) - 1}{ka I_1(ka)} \right\} .$$

$$. \quad . \quad . \quad (3.3)$$

The meridian sections are plotted in fig. 2 (a) for various values of ka, while the variation in shape is shown in fig. 2 (b) by re-plotting the curves of fig. 2 (a), making the central deflexions equal. It is not possible to compare with the membrane in this case, since the deflexion of a membrane is logarithmically infinite at a concentrated load.

The deflexion at the centre is

$$\frac{P}{2\pi T} \left\{ \log a + \gamma - \log 2 + \frac{2 - I_0(ka) - ka K_1(ka)}{ka I_1(ka)} \right\} , \quad (3.4)$$

$\gamma$ being Euler's constant, $0{\cdot}57721 \ldots$ .

### 4. *Load uniformly distributed round a Ring; Green's Function for Symmetrical Loading.*

We will next obtain the deflexion due to a load uniformly distributed round a ring of radius $b(<a)$ concentric with the boundary, which, if the total load is unity, gives Green's function for symmetrical loading.

The solution of the fundamental equation appropriate to the inner area ($r < b$, denoted by suffix $i$) is, taking account of the finiteness of the deflexion and absence of load at the origin,

$$w_i = C_i + D_i I_0(kr). \quad . \quad . \quad . \quad (4.11)$$

For the outer ring ($b < r < a$, denoted by suffix $e$) the formula is

$$w_e = A_e \log r + C_e + D_e I_0(kr) + F_e K_0(kr). \quad . \quad (4.12)$$

The conditions to be satisfied when $r = b$ are :

(1) continuity of displacement, i. e., $w_i = w_e$,

or $\qquad C_i + D_i I_0(kb) = A_e \log b + C_e + D_e I_0(kb) + F_e K_0(kb) :$

$$. \quad . \quad . \quad (4.21)$$

(2) continuity of slope, i. e., $dw_i/dr = dw_e/dr$,

or $\qquad D_i I_0'(kb) = A_e/kb + D_e I_0'(kb) + F_e K_0'(kb) ; \quad . \quad (4.22)$

(3) continuity of the bending moment, G,

i. e., $\qquad d^2w_i/dr^2 + \sigma/r \, . \, dw_i/dr = d^2w_e/dr^2 + \sigma/r \, . \, dw_e/dr,$

or $\qquad D_1 \{ I_0''(kb) + \sigma/kr \, . \, I'(kb) \} = - A_e(1 - \sigma)/k^2b^2$

$$+ D_e \{ I''(kb) + \sigma/kr \, . \, I'(kb) \}$$

$$+ F_e \{ K_0''(kb) + \sigma/kr \, . \, K_0'(kb) \}. \quad . \quad (4.23)$$

Using the differential equation satisfied by $I_0$ and $K_0$, after multiplying (4.22) by $(1 - \sigma)/kb$ and adding to (4.23), we have

$$D_i I_0(kb) = D_e I_0(kb) + F_e K_0(kb). \quad . \quad . \quad (4.24)$$

We also have the further condition

$$\underset{\epsilon \to 0}{\mathrm{L}^t} \left[ \left\{ 2\pi r \frac{\partial}{\partial r} (\nabla^2 - k^2) w_e \right\}_{r=b+\epsilon} \right.$$
$$\left. - \left\{ 2\pi r \frac{\partial}{\partial r} (\nabla^2 - k^2) w_i \right\}_{r=b-\epsilon} = 1, \quad (4.25)$$

whence

$$A_e = -\frac{1}{2\pi T}. \quad . \quad . \quad . \quad . \quad (4.31)$$

When $r = a$, we have

$$w_e = 0, \quad \text{or} \quad A_e \log a + C_e + D_e I_0(ka) + F_e K_0(ka) = 0, \quad (4.26)$$

$$dw_e/dr = 0, \quad \text{or} \quad A_e/ka + D_e I_0'(ka) + F_e K_0'(ka) = 0. \quad (4.27)$$

Solving these five equations, (4.21), (4.22), (4.24), (4.26), and (4.27), for the remaining five constants, we find

$$C_i = \frac{1}{2\pi T} \left\{ \log \frac{a}{b} - \frac{I_0(ka) - I_0(kb)}{ka I_1(ka)} \right\}, \quad . \quad . \quad . \quad (4.32)$$

$$D_i = \frac{1}{2\pi T} \left\{ \frac{1}{ka I_1(ka)} - K_0(kb) - \frac{K_1(ka) I_0(kb)}{I_1(kb)} \right\}, \quad (4.31)$$

$$C_e = \frac{1}{2\pi T} \left\{ \log a - \frac{I_0(kb) - I_0(kb)}{ka I_1(ka)} \right\}, \quad . \quad . \quad . \quad (4.34)$$

$$D_e = \frac{1}{2\pi T} \left\{ \frac{1}{ka I_1(ka)} - \frac{K_1(ka) I_0(kb)}{I_1(ka)} \right\}, \quad . \quad . \quad (4.35)$$

$$F_e = -\frac{I_0(kb)}{2\pi T}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4.36)$$

Thus

$$w_i = \frac{1}{2\pi T} \left[ \log \frac{a}{b} - \frac{I_0(ka) - I_0(kb)}{ka I_1(ka)} \right.$$
$$\left. + I_0(kr) \left\{ \frac{1}{ka I_1(ka)} - K_0(kb) \frac{K_1(ka) I_0 kb}{I_1(ka)} \right\} \right], \quad (4.41)$$

$$w_e = \frac{1}{2\pi T} \left[ \log \frac{a}{r} - \frac{I_0(ka) - I_0(kb)}{ka I_1(ka)} \right.$$
$$\left. + I_0(kr) \left\{ \frac{1}{ka I_1(ka)} - \frac{K_1(ka) I_0(kb)}{I_1(ka)} \right\} - K_0(kr) I_0(kb) \right].$$
$$. \quad . \quad . \quad (4.42)$$

It is to be noted that the complete function is symmetrical in $b$ and $r$, i. e., interchanging $b$ and $r$ interchanges $w_e$ and $w_i$. It is also easily verified that putting $b=0$ in the formula for $w_e$ gives essentially (3.3).

## 5. *Uniform Pressure over a Circle concentric with the Boundary.*

The result of § 4 can be used to find the deflexion due to any symmetrical pressure distribution, but we shall content ourselves by giving the results for a uniform pressure over a circle of radius $c(<a)$. When $r>c$,

$$w = \int_0^c 2\pi b p w_e \, db, \quad . \quad . \quad . \quad . \quad (5.1)$$

and the integral is easily evaluated by means of the formulæ

$$\int_0^c b \mathrm{I}_0(kb) \, db = \frac{c}{k} \mathrm{I}_0'(kc), \quad . \quad . \quad . \quad (5.21)$$

$$\int_0^c b \mathrm{K}_0(kb) \, db = \frac{c}{k} \mathrm{K}_0'(kc). \quad . \quad . \quad . \quad (5.22)$$

Thus we find

$$w = \frac{pc^2}{2\,\Gamma} \left[ \log \frac{a}{r} - \frac{\mathrm{I}_0(ka)}{ka\mathrm{I}_1(ka)} + \frac{2\mathrm{I}_1(kc)}{k^2 ac\mathrm{I}_1(ka)} \right.$$
$$+ \mathrm{I}_0(kr) \left\{ \frac{1}{ka\mathrm{I}_1(ka)} - \frac{2\mathrm{K}_1(ka)\mathrm{I}_1(kc)}{kc\mathrm{I}_1(ka)} \right\}$$
$$\left. - \frac{2\mathrm{K}_0(kr)\mathrm{I}_1(kb)}{kb} \right]. \quad . \quad . \quad . \quad . \quad (5.31)$$

When $r<c$, we have

$$w = \int_0^r 2\pi b p w_e \, db + \int_r^c 2\pi b p w_i \, db$$
$$= \frac{pc^2}{2\mathrm{T}} \left[ \frac{1}{2}\left(1 - \frac{r^2}{c^2}\right) + \log \frac{a}{c} - \frac{\mathrm{I}_0(ka)}{ka\mathrm{I}_1(ka)} + \frac{2\mathrm{I}_1(kc)}{k^2 ac\mathrm{I}_1(ka)} - \frac{2}{k^2 c^2} \right.$$
$$\left. + \mathrm{I}_0(kr) \left\{ \frac{1}{ka\mathrm{I}_1(ka)} + \frac{2\mathrm{K}_1(kc)}{kc} - \frac{2\mathrm{K}_1(ka)\mathrm{I}_1(kc)}{kc\mathrm{I}_1(ka)} \right\} \right].$$
$$. \quad . \quad . \quad (5.32)$$

### 6. *Concentrated Load at an Arbitrary Point;*
### *the general Green's Function.*

We next consider the deflexion due to a concentrated load at any point of the plate, which is equivalent to finding the general Green's function. By means of this, the deflexion due to any distribution of normal load can be found in the usual manner.

We imagine a load P applied at the point whose polar coordinates are $(c, 0)$, $c < a$. (If the coordinates are required as $(c, \alpha)$ we merely write $\theta - \alpha$ for $\theta$ in the solution below.) If the distance of the point $(r, \theta)$ from the point $(c, 0)$ be denoted by $\rho$ (confusion with other significances of $\rho$ is not likely), we have

$$\rho^2 = r^2 + c^2 - 2cr \cos \theta. \quad . \quad . \quad . \quad (6.1)$$

The analysis of § 3 shows that, in the neighbourhood of $(c, 0)$, the deflexion must contain terms of the form

$$w_1 = -\frac{2\pi T}{P} \{\log \rho + K_0(k\rho)\}. \quad . \quad . \quad (6.2)$$

To this must be added, in order to satisfy the boundary conditions, the general solution of the fundamental equation which is finite, and has zero load, at the origin, *i. e.*,

$$w_2 = C_0 + \sum_{n=1}^{\infty} C_n r^n \cos n\theta + \sum_{n=1}^{\infty} D_n I_n(kr) \cos n\theta, \quad (6.3)$$

the terms involving sines of multiples of $\theta$ being absent owing to symmetry about the initial line.

The expression in (6.1) may be expanded in terms of functions of $r$ and $\theta$ by means of well-known expansions :—

$r < c$,

$$w_1 = -\frac{1}{2\pi T} \left\{ \log c - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{r}{c}\right)^n \cos n\theta + I_0(kr) K_0(kc) \right.$$
$$\left. + 2 \sum_{n=1}^{\infty} I_n(kr) K_n(kc) \cos n\theta \right\} ; \quad (6.41)$$

$r > c$,

$$w_1 = -\frac{1}{2\pi T} \left\{ \log r - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{c}{r}\right)^n \cos n\theta + K_0(kr) I_0(kc) \right.$$
$$\left. + 2 \sum_{n=1}^{\infty} K_n(kr) I_n(kc) \cos n\theta \right\}. \quad (6.42)$$

The boundary conditions at $r = a$ necessitate

(i.) $w_1 + w_2 = 0$, and since coefficients of cosines of multiples of $\theta$ must vanish independently, we have

$$\left.\begin{aligned}
C_0 + D_0 I_0(ka) &= \frac{1}{2\pi T} \left\{ \log a + I_0(kc) K_0(ka) \right\}, \\
C_n a^n + D_n I_n(ka) &= \frac{1}{2\pi T} \left\{ -\frac{1}{n}\left(\frac{c}{a}\right)^n + 2 I_n(kc) K_n(ka) \right\} ;
\end{aligned}\right\}$$

$$. \quad . \quad . \quad (6.51)$$

(ii.) $\dfrac{d}{dr}(w_1 + w_2) = 0$, whence

$$\left.\begin{aligned}
D_0 ka I_0'(ka) &= \frac{1}{2\pi T} \left\{ \frac{1}{ka} + ka I_0(kc) K_0'(ka) \right\}, \\
n C^n a^n + D_n ka I_n'(ka) &= \frac{1}{2\pi T} \left\{ \left(\frac{c}{a}\right)^n + 2ka I_n(kc) K_n'(ka) \right\}.
\end{aligned}\right\}$$

$$. \quad . \quad . \quad (6.52)$$

Solving these equations, making use of the formulæ

$$n I_n(z) - z I_n'(z) = -z I_{n+1}(z),$$
$$n K_n(z) - z K_n'(z) = -z K_{n+1}(z),$$
$$K_n(z) I_n'(z) - K_n'(z) I_n(z) = (-)^n z^{-1},$$

we find

$$C_0 = \frac{1}{2\pi T} \left\{ \log a - \frac{I_0(ka) - I_0(kc)}{ka I_1(ka)} \right\},$$

$$D_0 = \frac{1}{2\pi T} \frac{1 - ka K_1(ka) I_0(kc)}{ka I_1(ka)},$$

$$C_n = \frac{1}{2\pi T a^n} \left\{ -\frac{1}{n}\left(\frac{c}{a}\right)^n - (-)^n \frac{\{ 2 I_n(ka) - I_n(kc) \}}{ka I_{n+1}(ka)} \right\},$$

$$D_n = \frac{2}{2\pi T} \frac{(c/a)^n + I_n(kc) K_{n+1}(ka)}{ka I_{n+1}(ka)}.$$

## 7. *Vibrations.*

The free vibrations of a clamped circular disk in tension are determined from the differential equation

$$B \nabla^2 (\nabla^2 - k^2) w = -2\rho h \ddot{w}. \quad . \quad . \quad (7.11)$$

If $\omega$ is the pulsatance, so that $w \propto \sin(\omega t - \epsilon)$, the differential equation for the amplitude is

$$B\nabla^2(\nabla^2 - k^2)w = 2\rho h\omega^2 w, \quad . \quad . \quad . \quad (7.12)$$

or
$$\left(\nabla^4 - k^2\nabla^2 - \frac{2\rho h\omega^2}{B}\right)w = 0. \quad . \quad . \quad . \quad . \quad (7.13)$$

This may be factorized as

$$(\nabla^2 + \alpha^2)(\nabla^2 - \beta^2)w = 0, \quad . \quad . \quad . \quad (7.14)$$

where
$$\left.\begin{array}{c} \beta^2 - \alpha^2 = k^2 = T/B, \\ \alpha^2\beta^2 = 2\rho h\omega^2/B. \end{array}\right\} \quad . \quad . \quad . \quad (7.15)$$

The general solution of (7.14) is

$$\sum_{n=0}^{\infty} \left[ \{ A_n J_n(\alpha r) + C_n Y_n(\alpha r) + D_n I_n(\beta r) + F_n K_n(\beta r) \} \cos n\theta \right.$$
$$\left. + \{ A_n' J_n(\alpha r) + C_n' Y_n(\alpha r) + D_n' I_n(\beta r) + F_n' K_n(\beta r) \} \sin n\theta \right].$$
$$\quad . \quad . \quad (7.21)$$

If the plate is complete, the conditions at the origin necessitate the absence of the terms in $Y_n$ and $K_n$.

## 8. Symmetrical Modes.

For the symmetrical modes, we need only terms of zero order, and so take

$$w = A_0 J_0(\alpha r) + D_0 I_0(\beta r). \quad . \quad . \quad . \quad (8.1)$$

For a clamped edge at $r = a$ we must have

$$w = 0, \quad i.e., \qquad A_0 J_0(\alpha a) + D_0 I_0(\beta a) = 0; \quad . \quad (8.21)$$

$$dw/dr = 0, \quad i.e., \quad A_0\alpha J_0'(\alpha a) + D_0\beta I_0(\beta a) = 0. \quad . \quad (8.22)$$

These are compatible, with non-vanishing constants, only if

$$\frac{\alpha J_0'(\alpha a)}{J_0(\alpha a)} = \frac{\beta I_0'(\beta a)}{I_0(\beta a)}, \quad . \quad . \quad . \quad (8.31)$$

or
$$\frac{\alpha J_1(\alpha a)}{J_0(\alpha a)} + \frac{\beta I_1(\beta a)}{I_0(\beta a)} = 0, \quad . \quad . \quad (8.32)$$

and, with $\alpha$ (and $\beta$) determined as a root of this equation, the deflected form is given by

$$w \propto \left\{ \frac{J_0(\alpha r)}{J_0(\alpha a)} - \frac{I_0(\beta r)}{I_0(\beta a)} \right\} \quad . \quad . \quad . \quad (8.4)$$

For convenience of solution, we write the equation (8.32), with the help of the first of (7.15) and with the abbreviations $\alpha a = \phi$, $ka = \gamma$,

$$\frac{\phi J_1(\phi)}{J_0(\phi)} = - \frac{\sqrt{(\phi^2 + \gamma^2)} \cdot I_1 \sqrt{(\phi^2 + \gamma^2)}}{I_0 \sqrt{(\phi^2 + \gamma^2)}}. \quad . \quad (8.5)$$

The roots of this equation have been determined for several values of $\gamma$, and are given in Table I. From a value of $\phi$, we obtain, using (7.15),

$$\omega^2 = \phi^2(\phi^2 + \gamma^2) \; B/2\rho h a^4. \quad . \quad . \quad . \quad (8.6)$$

The values of $\phi^2(\phi^2 + \gamma^2)$ are also tabulated.

The ratios of the radii of the nodal circles necessitates the solution for $r$ of the equation

$$\frac{J_0(\alpha r)}{J_0(\alpha a)} = \frac{I_0(\beta r)}{I_0(\beta a)},$$

for the values of $\alpha$ and $\beta$ corresponding to the $\phi$'s of Table I. Their probable importance does not, however, seem to warrant the labour of computing them.

There are two known methods of obtaining an approximate value of the pulsatance or frequency of the fundamental mode. The first is the use of Rayleigh's Principle, which is equivalent to equating the mean kinetic and potential energy for an assumed mode of vibration. It is known that the approximation always gives too great frequencies. If we use the deflexion formula for uniform pressure, in the absence of tension,

$$w \propto (a^2 - r^2)^2, \quad . \quad . \quad . \quad . \quad (8.71)$$

we find that

$$\omega^2 = \frac{320}{3} \frac{B}{2\rho h a^2} \left\{ 1 + \frac{Ta^2}{16B} \right\} . . \quad . \quad (8.72)$$

When T is zero $(ka = 0)$ this gives $106 \cdot 67$ as the numerical factor instead of $104 \cdot 36$, an error of over 2 per cent. in $\omega^2$. When B is zero, *i. e.*, for a membrane $(ka = \infty)$, it gives

$$\omega^2 = \frac{20}{3} \frac{T}{2\rho h a^2}, \quad . \quad . \quad . \quad . \quad (8.73)$$

with a numerical factor of $6 \cdot 67$ instead of $5 \cdot 78$, an error of about 17 per cent. (in $\omega^2$). For intermediate values of $ka$, the percentage error does not steadily increase, but becomes quite small in the neighbourhood of $ka = 5$. Values of the numerical factor in (8.72) comparable with $\phi^2(\phi^2 + \gamma^2)$, are

Dr. W. G. Bickley *on Deflexions and*

given in Table I.   The method of successive approximation [*]
could be used to improve these approximations, but the form

## TABLE I.

### Roots of Equation (8.5) and Derived Numbers.

| $\gamma$. | $\phi$. | $\phi^2(\phi^2+\gamma^2)$. | Southwell approx. | Rayleigh approx. |
|---|---|---|---|---|
| **First Root—Fundamental Mode.** | | | | |
| 0 ............... | 3·1962 | 104·36 | 104·36 | 106·67 |
| 1 ............... | 3·1719 | 111·35 | 110·14 | 113·33 |
| 2 ............... | 3·1079 | 131·94 | 127·49 | 133·33 |
| 5 ............... | 2·8674 | 273·14 | 248·94 | 273·33 |
| 10 ............... | 2·6562 | 755·34 | 682·67 | 773·33 |
| 20 ............... | 2·5197 | 2579·8 | 2417·6 | 2773·3 |
| $\infty$ ............... | 2·4048 | | $\rightarrow 5\cdot7832\,\gamma^2$ | $\rightarrow 20\,\gamma^2/3$ |
| **Second Root.** | | | | |
| 0 ...... ........ | 6·3065 | 1581·8 | 1581·8 | |
| 1 ............... | 6·3002 | 1615·2 | 1612·2 | |
| 2 ............... | 6·2822 | 1715·5 | 1703·4 | |
| 5 ............... | 6·1815 | 2415·3 | 2342·2 | |
| 10 ............... | 5·9953 | 4888·2 | 4623·4 | |
| 20 ............... | 5·7914 | 14541 | 13748 | |
| $\infty$ ............... | 5·5201 | | $\rightarrow 30\cdot416\,\gamma^2$ | |
| **Third Root.** | | | | |
| 0 ............... | 9·4395† | 7939·6 | 7939·6 | |
| 1 ............... | 9·4367 | 8019·3 | 8014·4 | |
| 2 ............... | 9·4285 | 8258·2 | 8239·1 | |
| 5 ............... | 9·3770₅ | 9929·7 | 9811·7 | |
| 10 ............... | 9·2504 | 15879 | 15428 | |
| 20 ............... | 9·0446 | 39414 | 37894 | |
| $\infty$ ............... | 8·6537 | | $\rightarrow 74\cdot887\,\gamma^2$ | |

† The value of this root is given by Jahnke and Emde as 9·425 (*i. e.*, $3\pi$).
The higher roots (for $\gamma=0$) approximate to multiples of $\pi$, and, except for the two lower roots, it is only these multiples that Jahnke and Emde give.   A similar remark applies to Jahnke |and Emde's values for other values of $n$, corresponding to Table III.

* See G. Temple, Proc. Roy. Soc. A, cxix. pp. 275–293 (1928), or a forthcoming book on Rayleigh's Principle by Temple and Bickley, Oxford U. P. (1933).

of the Green's function, (4.41) and (4.42), is such that transcendental functions (I's and K's) would enter the formulæ, and the labour would not be light, although the solution of transcendental equations would not be necessary, and, in fact, the process would involve no trial and error.

The other method, due to Southwell *, is analogous to that suggested by Dunkerley in his paper upon the whirling of shafts, but with the difference that in the case of shafts, the kinetic energy is in two parts (shaft + pulley) while the potential is in one (shaft), whereas in the present problem it is the potential energy which is compound, and the kinetic single. If $\omega_1$ is the pulsatance for one form of potential energy only (say, bending in the absence of tension), $\omega_2$ that for the other form (tension with a perfectly flexible plate), and $\omega$ the true pulsatance, then it is known, and can be proved by applying Rayleigh's Principle with the true amplitude formula to the approximate determination of $\omega_1$ and $\omega_2$, that

$$\omega_1{}^2 + \omega_2{}^2 \leq \omega^2, \quad . \quad . \quad . \quad . \quad . \quad (8.81)$$

*i. e.*, we are provided with a lower limit to the fundamental frequency. In our case

$$\left.\begin{array}{l} \omega_1{}^2 = 104\cdot36 \quad B/2\rho h a^4, \\ \omega_2{}^2 = \quad 5\cdot7832 \ T/2\rho h a^2, \end{array}\right\} \quad . \quad . \quad (8.82)$$

so that

$$\omega^2 \geq 104\cdot36\,\frac{B}{2\rho h a^4} + 5\cdot783\,\frac{T}{2\rho h a^2}$$

$$= \frac{B}{2\rho h a^4}\{104\cdot36 + 5\cdot783\,\gamma^2\}. \quad . \quad (8.83)$$

The values of the numerical constant in (8.83) are also given in Table I., and the two approximations can be compared.

It is seen that the true value always lies between the approximate ones. In their paper on "The Vibrations of a Spinning Disk," Lamb and Southwell † obtained, in a similar manner, upper and lower limits to the frequency of one mode for a particular disk, and were fortunate in obtaining very close limits in their one instance. Although the errors in Table I. are possibly greater than even engineers will

* Lamb and Southwell, " Vibrations of a Spinning Disk," Proc. Roy. Soc. A, xcix. (1921).
† *Loc. cit., ante.*

readily tolerate, yet the fact that in such problems upper
and lower limits are obtainable with comparative ease
deserves emphasis. In the intended application of the
present results, in fact, the lower limit is adequate, since it
is only necessary that the fundamental frequency shall not
be less than, say, $10,000 \sim$ /sec.

While Rayleigh's method applies only to the fundamental,
Southwell's applies to all modes. The percentage errors in
the frequency, for these approximations, are plotted in
fig. 3 *. It will be noticed that for the higher modes,
Southwell's approximation becomes progressively better, and
especially for the smaller values of $\gamma$.

Fig. 3.
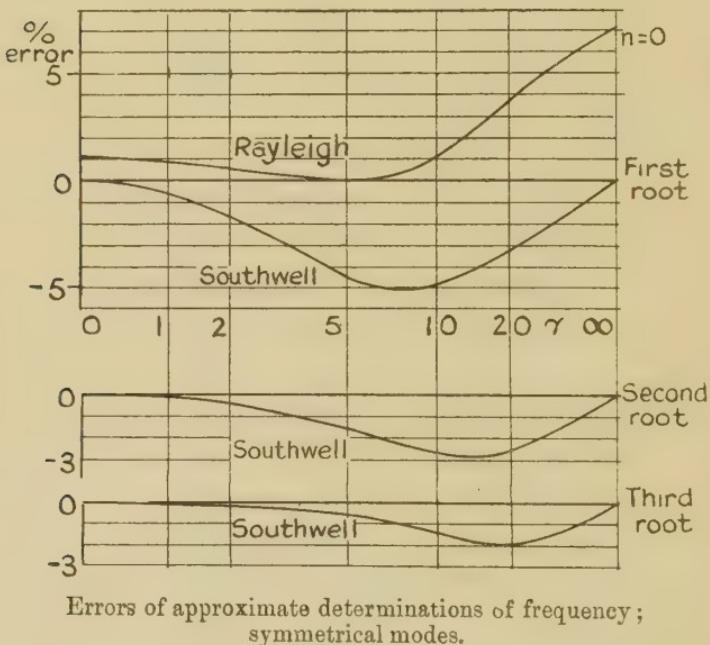


Errors of approximate determinations of frequency;
symmetrical modes.

## 9. *Modes with Nodal Diameters.*

If the deflexion in any mode has $n$ nodal diameters, its
variation along the radius must be given by the formula

$$w = A_n J_n(\alpha r) + B_n I_n(\beta r), \quad . \quad . \quad . \quad (9.1)$$

(provided the plate is complete, the $Y_n$ and $K_n$ terms being

* The abscissæ are given by $\gamma/(\gamma \div 5)$, which enables all values of $\gamma$ to
be exhibited.

then absent).   To satisfy the conditions at the edge, we must have

$$w = 0, \quad i.\,e\,, \qquad \mathrm{A}_n\mathrm{J}_n(\alpha a) + \mathrm{B}_n\mathrm{I}_n(\beta a) = 0\,;$$
$$dw/dr = 0, \quad i.\,e., \quad \alpha\mathrm{A}_n\mathrm{J}_n{}'(\alpha a) + \beta\mathrm{B}_n\mathrm{I}_n{}'(\beta a) = 0.\qquad (9.2)$$

The equation for the vibration frequencies is therefore

$$\frac{\alpha\mathrm{J}_n{}'(\alpha a)}{\mathrm{J}_n(\alpha a)} = \frac{\beta\mathrm{I}_n{}'(\beta a)}{\mathrm{I}_n(\beta a)}. \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad (9.3)$$

But
$$n\mathrm{I}_n(\beta a) - \beta a\mathrm{I}_n{}'(\beta a) = -\beta a\mathrm{I}_{n+1}(\beta a),$$
$$n\mathrm{J}_n(\alpha a) - \alpha a\mathrm{J}_n{}'(\alpha a) = +\alpha a\mathrm{J}_{n+1}(\alpha a),$$

so that this reduces to

$$\frac{\alpha\mathrm{J}_{n+1}(\alpha a)}{\mathrm{J}_n(\alpha a)} + \frac{\beta\mathrm{I}_n{}'(\beta a)}{\mathrm{I}_n(\beta a)} = 0. \quad \cdot \quad \cdot \quad \cdot \quad (9.41)$$

In our previous notation, this is

$$\frac{\phi\mathrm{J}_{n+1}(\phi)}{\mathrm{J}_n(\phi)} = - \frac{\surd(\phi^2 + \gamma^2) \cdot \mathrm{I}_{n+1}\surd(\phi^2 + \gamma^2)}{\mathrm{I}_n\surd(\phi^2 + \gamma^2)}. \quad (9.42)$$

A few values of the roots of this equation, for the same set of values of $\gamma$ as used before, are given in Table II., together with the numerical constants deduced therefrom which lead to a determination of the natural frequencies. The labour of obtaining these roots has been rather heavy. This was not due so much to the actual nature of equation (9.42), as to the absence of sufficiently "fine-grained" tables of Bessel functions of order higher than unity. Consequently, as a preliminary, tables of $\mathrm{J}_2$ and $\mathrm{J}_3$ at interval $0\cdot01$ had to be prepared, over the necessary ranges of the argument.   Meissel's tables of $\mathrm{J}_0$ and $\mathrm{J}_1$ (given by Gray and Mathews), which made dealings with equation (9.5) comparatively speedy, formed the starting-point.   Values of $\mathrm{J}_2$ and $\mathrm{J}_3$ were calculated at interval $0\cdot1$ (except where found in Watson's 'Bessel Functions'), by means of the recurrence formulæ, correct to seven places of decimals, and these were then interpolated to tenths by Comrie's "end figure" process *.   The method was a joy to use—at least until its novelty wore off,—and the way in which the calculated values were reproduced after every tenth step almost uncannily miraculous.   Indeed, it was far speedier and less laborious than using the recurrence formula throughout. The other side of equation (9.42) is more tractable, since for

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* 'Nautical Almanack' for 1931.   The pages dealing with this process, and the necessary tables, are reprinted in pamphlet form. (H.M. Stationery Office.)

moderate values of the argument, $I_{n+1}/I_n$ approaches monotonically its limit unity, and the asymptotic expansion of this ratio makes extension beyond the range of existing tables a simple matter, especially as variation is slow, and thus a wide interval can be used.

The values for $\gamma = \infty$, *i. e.*, roots at the Bessel functions $J_n$, were taken from Watson.

Rayleigh's Principle, using the simplest algebraic formula satisfying the boundary conditions, *i. e.*,

$$w = Cr^n \cos n\theta (a^2 - r^2)^2, \quad . \quad . \quad . \quad . \quad (9.5)$$

Fig. 4.
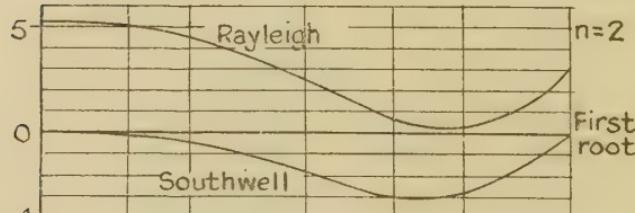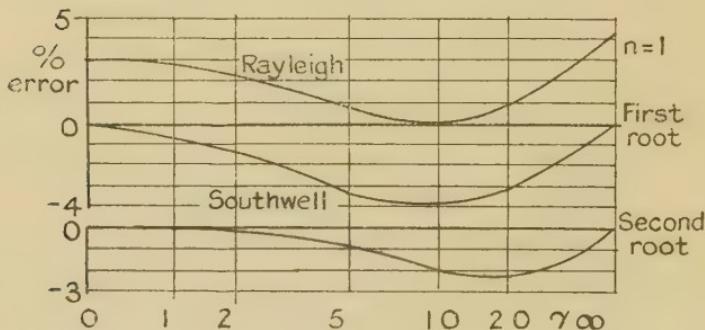


Errors of approximate determinations of frequency; modes with nodal diameters.

leads to the approximation to the slowest pulsatance with $n$ nodal diameters,

$$\omega^2 = \frac{8(n+1)(n+2)(n+4)(n+5)}{3} \frac{B}{2\rho ha^4} \left\{ 1 + \frac{\gamma^2}{2(n+2)(n+4)} \right\}.$$
$$. \quad . \quad . \quad (9.6)$$

Values of the numerical factor of $B/2\rho ha^4$ derived from this are also given in Table II., along with the Southwell approximation. The percentage errors of these approximations are exhibited in fig. 4. These approximations exhibit the same features as in the case of $n = 0$.

Finally, in Table III. we give the numerical values of the frequencies, in cycles per second, for the diaphragm whose

## TABLE II.

| $\gamma$. | $\phi$. | $\phi^2(\phi^2+\gamma^2)$. | Southwell approx. | Rayleigh approx. |
|---|---|---|---|---|
| **First Root.** | | $n = 1.$ | | |
| 0 ............... | 4·6109 | 452·01 | 452·01 | 480 |
| 1 ............... | 4·5997 | 468·80 | 466·69 | 496 |
| 2 ............... | 4·5684 | 519·06 | 510·74 | 544 |
| 5 ............... | 4·4148 | 867·14 | 819·06 | 880 |
| 10 ............... | 4·2007$_5$ | 2076·0 | 1920·2 | 2080 |
| 20 ............... | 4·0265 | 6747·9 | 6324·8 | 6880 |
| $\infty$ ............... | 3·8317 | | $\to 14{\cdot}682\,\gamma^2$ | $\to 16\,\gamma^2$ |
| **Second Root.** | | | | |
| 0 ............... | 7·7993 | 3700·1 | 3700·1 | |
| 1 ............... | 7·7952 | 3753·2 | 3749·2 | |
| 2 ............... | 7·7834 | 3912·5 | 3897·0 | |
| 5 ............... | 7·7126 | 5025·5 | 4930·6 | |
| 10 ............... | 7·5577 | 8974·3 | 8622·0 | |
| 20 ............... | 7·3476 | 24488 | 23388 | |
| $\infty$ ............... | 7·0156 | | $\to 49{\cdot}218\,\gamma^2$ | |
| **First Root.** | | $n = 2.$ | | |
| 0 ............... | 5·9057 | 1216·4 | 1216·4 | 1344 |
| 1 ............... | 5·8991 | 1245·8 | 1242·8 | 1372 |
| 2 ............... | 5·8808 | 1334·4 | 1321·9 | 1456 |
| 5 ............... | 5·7752 | 1946·3 | 1875·8 | 2044 |
| 10 ............... | 5·5867 | 4095·2 | 3253·9 | 4144 |
| 20 ............... | 5·3898 | 12464 | 11766 | 12544 |
| $\infty$ ............... | 5·1356 | | $\to 26{\cdot}375\,\gamma^2$ | $\to 28\,\gamma^2$ |

dimensions are given in § 2 above, as calculated from the above results, and fig. 5 is a graph showing the dependence of the six frequencies upon $\gamma$ (*i. e.*, upon the tension).

## TABLE III.

### Vibration Frequencies for Sample Disk.

| No. of Nodal circles | 0. | 1. | 2. | 0. | 1. | 0. | T |
| ,, Nodal diam. | 0. | 0. | 0. | 1. | 1. | 2. | dynes/cm. |
| $\gamma$. | Frequency ($\sim$/sec.). | | | | | | |
| 0 ......... | 1247 | 4855 | 10876 | 2595 | 7425 | 4257 | 0 |
| 1 ......... | 1288 | 4906 | 10931 | 2643 | 7478 | 4308 | $3 \cdot 623 \times 10^5$ |
| 2 ... ..... | 1402 | 5056 | 11104 | 2781 | 7635 | 4459 | $1 \cdot 449 \times 10^6$ |
| 5 ......... | 2017 | 5999 | 12163 | 3594 | 8653 | 5385 | $9 \cdot 057 \times 10^6$ |
| 10 ......... | 3355 | 8532 | 15430 | 5562 | 11562 | 7811 | $3 \cdot 623 \times 10^7$ |
| 20 ......... | 6200 | 14719 | 24238 | 10027 | 19101 | 13627 | $1 \cdot 449 \times 10^6$ |

Fig. 5.
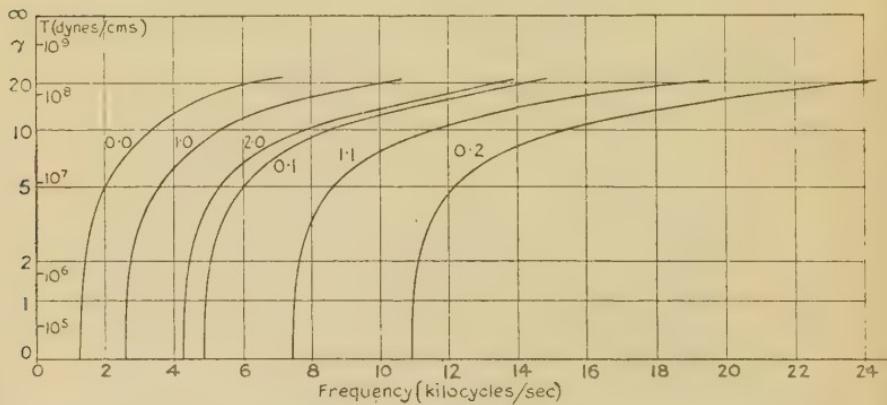


Dependence of frequency upon tension. The numbers attached to the curves denote the numbers of nodal circles and diameters, respectively.

## Summary.

In view of the proposed design of a condenser microphone, in which the natural frequencies of vibration of the diaphragm were to be raised by the application of tension in its plane, the investigation of the effects of this tension upon the normal displacement under pressure, and upon the natural frequencies of vibration, was undertaken. The present paper gives the results of this investigation.

The displacements due to several types of normal loading are worked out, and the effects of variation of the tension upon the deflected form are exhibited by diagrams. The general Green's function for normal load is also given.

The equations, upon the solutions of which the calculation of the frequencies of vibration of the various normal modes depends, are given, and solved numerically in the case of six modes, for a range of values of the tension. Comparison with two approximate methods, Rayleigh's and Southwell's, which enable upper and lower limits to the frequencies to be determined, is made. Finally, numerical values of the frequencies for a sample diaphragm are calculated.

Oct. 31, 1932.

## LXIII. *Theory of Diamagnetism for Mono- and Polynuclear Systems. By* D. P. Ray-Chaudhuri, *Calcutta* *.

### 1. *Application to Mononuclear Systems.*

THE theory of diamagnetism as developed up till now applies only to isolated mononuclear systems. The susceptibility per gram-atom or gram-ion of a diamagnetic substance may be expressed by the relation [1].

$$\chi = -\frac{Ne^2}{6mc^2} \Sigma \bar{r}_\kappa^2 + \frac{2}{3} \frac{N}{h} \underset{n}{\Sigma}' \frac{|M(n', n)|^2}{\nu(n', n)}. \quad . \quad . \quad (1)$$

where

$N =$ Avogadro's number.

$\bar{r}_\kappa^2 =$ Time average value of the square of the orbital radius of the $\kappa$th electron, the summation extending over all electrons in the system.

$M(n', n) =$ a non-diagonal element of the matrix for the angular momentum of the system.

$\nu(n', n) =$ frequency corresponding to $n'$, $n$ transition. Other symbols have their usual significance.

The second term on the right-hand side of (1) arises from the part of the magnetic moment perpendicular to the resultant angular momentum and vanishes for states in which both $l$ and $s$ are zero ; *i. e.*, for ¹S state of the atom or the ion. We need not consider systems in other than this state, as their paramagnetism will far outweigh the diamagnetism.

With this restriction it is possible to calculate theoretically the value of $\chi$, as then the only unknown quantity

* Communicated by Prof. D. M. Bose, M.A., Ph.D.

on the right side of (1) is $r_\kappa^2$, the value of which can variously be estimated. For Coulomb central fields wave mechanics [2] gives

$$r^2 = a_0^2 \frac{n^2}{(Z-s)^2} \left\{ \frac{5}{2} n^2 - \frac{3l(l+1)-1}{2} \right\}, \quad . \quad . \quad (2)$$

where $a_0$ = radius of the innermost hydrogen orbit.

  $n$ = principal quantum number.

  $l$ = orbital quantum number.

  $Z$ = nuclear charge.

  $s$ = screening constant.

The actual orbits are sought to be replaced by hydrogen-like ones, the central Coulomb field proceeding from not the real nuclear charge $Z$, but from an effective one $Z-s$, where $s$ accounts for the interactions of the electrons in an approximate way.

Under the additional assumption that nodes in the eigenfunction exert no appreciable influence on charge distribution Slater [3] obtains for $r^2$ the value

$$\bar{r^2} = a_0^2 \frac{(n^*)^2 (n^* + \frac{1}{2})(n^* + 1)}{(Z-s)^2} \quad . \quad . \quad . \quad (3)$$

where $n^*$ is an effective principal quantum number whose values for real values of $n$ have been given by Slater. While this neglect of nodes is justified for normal atoms or negative ions, they become of increasing importance as the positive charge on the ion increases. Application of Slater's method is, therefore, less justified in the case of multiply-charged positive ions.

We may still mention two more methods of evaluating $r^2$. One is Hartree's [4], and the other is by Thomas and Fermi [5]. Hartree's method is equivalent to taking the product of the electronic wave-functions as the atomic wave-function, and takes no cognisance of resonance, the effect of considering which is to reduce diamagnetism by lowering the charge density. Hartree's values for diamagnetic susceptibility are thus always too high. The Thomas-Fermi statistics is applicable only to heavy atoms, and gives [6]

$$\chi \cdot a \cdot Z^{-1/3} = \text{constant}; \text{ where}$$

$a$ = atomic volume.

By a very rough approximation this also gives

$$\chi_M = -10^{-5} \cdot Z^{1/3}.$$

This should not be expected to agree with experiment as their statistics applies only to the inner electrons of heavy atoms, while diamagnetism gets most of its contribution from outer electrons.

As may be seen from the following table, formulæ (2) and (3) give different estimates of the value of $r^2$. The values of the numerators in (2) and (3) are given.

TABLE I.

| $n$. | $l$. | $n^*$. | $n^2 \left\{ \dfrac{5}{2} n^2 - \dfrac{3l(l+1)-1}{2} \right\}$. | $(n^*)^2(n^*+\frac{1}{2})(n^*+1)$. |
|:---:|:---:|:---:|:---:|:---:|
| 1 | 0 | 1 | 3 | 3 |
| 2 | 0 | 2 | 42 | 30 |
|   | 1 |   | 30 | 30 |
| 3 | 0 | 3 | 207 | 126 |
|   | 1 |   | 180 | 126 |
| 4 | 0 | 3·7 | 648 | 269·8 |
|   | 1 |   | 600 | 269·8 |
| 5 | 0 | 4·0 | 1575 | 360 |
|   | 1 |   | 1500 | 360 |

The table immediately shows how greatly divergent values may be obtained by using the different relations. In those cases where comparison with experimental values is possible, it is found that Slater's values are nearer to truth.

Screening constants also present a difficulty. Numbers denoting these are essentially parameters determining an average value of the electrostatic field in which the electron moves. There will, therefore, be as many different ones as there are ways of averaging, which means that for different physical processes involved the screening constant will be different. This is illustrated in the following table, which gives the different screening constants for helium.

There are two sets of theoretical screening constants available at present—one calculated in a half-classical way by Pauling [2], and the other obtained in a semi-empiric way by Slater [3]. Pauling introduces the refinement of designing different constants for different physical properties, but his method of calculation is classical,

quantum considerations being sought to be introduced by replacing $l^2$ by $l(l+1)$.

Slater's scheme is a tentative extension to heavier atoms of the relations found by Zener [7] to hold with a good approximation for light atoms, viz., Li to Ne. The set may be called " energy screening constants," giving the total energy of an atom by an expression

## TABLE II.

### Different Screening Constants for Helium.

| Dielectric constant. | Diamagnetic susceptibility. | Ionization potential. | Wave mechanical calculation for energy. | Slater's value. |
|---|---|---|---|---|
| 0·41 | 0·392 | 0·296 | 0·3125 | 0·30 |

## TABLE III.

### Ionization Potentials from Slater's Screening Constants in Hydrogen Units ; 1 unit = 13·54 volts.

|       | Calc. | Obs.  |        | Calc. | Obs.   |        | Calc. | Obs.  |
|-------|-------|-------|--------|-------|--------|--------|-------|-------|
| He ... | 1·78  | 1·808 | Na .. | ·537  | ·378   | K  .. | ·354  | ·319  |
| Li ... | ·423  | ·397  | Mg.. | ·610  | ·562   | Ca .. | ·439  | ·450  |
| Be ... | ·579  | ·701  | Al .. | ·793  | ·440   | Rb .. | ·302  | ·307  |
| B .... | ·718  | ·616  | Si .. | ·902  | ·60    | Sr .. | ·375  | ·419  |
| C .... | ·840  | ·838  | P  .. | 1·012 | ·761   | Cs .. | ·274  | ·287  |
| N .... | ·957  | 1·07  |        |       |        |        |       |       |
| O .... | 1·03  | 1·00  | S  .. | 1·115 | ·7614  |        |       |       |
|        |       |       |        |       |        | Ba .. | ·340  | ·382  |
| F .... | 1·12  | 1·37  | Cl .: | 1·194 | ·9677  |        |       |       |
| Ne ... | 1·17  | 1·587 | A  .. | 1·27  | 1·145  |        |       |       |

of the type $\Sigma \left(\dfrac{Z-s}{n^*}\right)^2$, the summation being over all

the electrons in the atom. The energy of removal of any electron or electrons may be obtained with fair accuracy from these values. Table III. gives the values of the ionization potentials of some elements as calculated from Slater's constants, and compares them with values experimentally obtained.

Both Pauling and Slater make *s* practically independent of Z. But the former depends largely on the latter, as has been shown by Saha [8] from X-ray data. He shows that for 1*s* electrons S can fairly be represented by the formula .

$$S = \cdot 895 + \cdot 1242 \, Z - \cdot 0014 \, Z^2 \text{ up to } Z = 20,$$

and

$$S = \cdot 895 + \cdot 124 \, (Z-1) - \cdot 0014 (Z-1)^2 \text{ for } Z = 20 \text{ to } 92.$$

For $2s$ electrons from Ne upwards

$$S = 7 \cdot 728 + \cdot 277 (Z-10) - \cdot 00231 \, (Z-10)^2.$$

The screening of outer electrons which is directly considered in Saha's scheme, is taken care of indirectly in Slater's. It seems probable on this account that the diamagnetism of inner electrons is actually greater than that given by Slater's scheme.

A comparison of susceptibility values calculated from Slater's data with the experimental values will be of great interest, as it is likely to bring out the degree of accuracy to be expected from theoretical computations ; but there is a difficulty. The inert gases offer the only sure ground for comparison. Experimental values for a number of ions in solution are available, but the fact that the $\chi$ value for a molecule may be, and has been, differently partitioned amongst the ions, leaves us uncertain as to which value to accept as true. For example, it may be stated that Ikenmeyer [9] took the ionic $\chi$'s to be inversely proportional to the squares of the nuclear charges. As most of the contribution to $\chi$ comes from the outer electrons, and as the behaviour of an electron is better represented by taking an effective nuclear charge, the proportionality ought to be given by the inverse square of an average effective nuclear charge, as has been suggested by Brindley [10]. The following table gives the $\chi$ values for a number of ions as obtained by different workers, as also the theoretically computed value of some. We are inclined to consider Hocart's [11] and Reicheneder's [12] measurements as very accurate. If the susceptibility of $H^+$ be taken as zero their measurements on HCl, HBr, and HI give directly the $\chi$'s for $Cl^-$, $Br^-$, and $I^-$. But Weiss [13] remarks that just as the optical refractivity of solutions is influenced by the effect of the charged ions on water molecules, so also the susceptibilities of salts in solution will be influenced and in a similar way. He estimates the magnitude of this effect, and finds the correction important for small ions like $H^+$, $Li^+$, $Mg^{++}$, $Ca^{++}$, but not so for large ones.

Angus [14] has introduced a modification in Slater's method of calculation in that while Slater takes the

$s$ and $p$ electrons together in calculating the screening, he introduces the refinement of taking them separately. This causes a lowering of the $\chi$ values by increasing the effective charges for the $s$ electrons, and the agreement is improved for heavier atoms where Slater's values are too high, while it is impaired for the lighter ones.

## TABLE IV.

### Comparison of some Computed and Observed $\chi$ values for some Ions.

|  |  | Cl⁻. | Br⁻. | I⁻. |
|---|---|---|---|---|
| Computed | Pauling ......... | 29·0 | 54 | 80 |
|  | Hartree ......... | 40·4 | — | — |
|  | Slater ........... | 25·8 | 40·1 | 59·8 |
|  | Angus ............ | 22·86 | 36·65 | 55·32 |
| Observed | Ikenmeyer ........ | 20·4 | 34·8 | 49·3 |
|  | Reicheneder ...... | 21·9 | 32·5 | 50·2 |
|  | Hocart ........... | 22·0 | — | — |
|  | Weiss............. | 23·1 | 34·1 | 50·2 |
|  | Abonnenc [15]*.... | 21·9 | 33·0 | 48·8 |

### The Inert Gases.

|  | Pauling. | Hartree. | Slater. | Angus. | Observed. |
|---|---|---|---|---|---|
| He ....... | 1·54 | 1·90 | 1·68 | 1·68 | 1·88 |
| Ne ....... | 5·7 | 8·81 | 5·72 | 5·07 | 6·66 |
| A ........ | 21·5 | 25·3 | 18·87 | 16·95 | 18·13 |
| Kr ....... | 42 | — | 31·73 | 29·33 | — |
| Xe ....... | 66 | — | 48·00 | 44·78 | — |

* If correction for $H^+$ be applied all the values will be increased by 1·1.

## 2. *Application to Molecules.*

In our application of the theory to molecules we generally assume that the diamagnetism of the atoms is conserved in the molecules, when equation (1) will be applicable with a modification in meaning. The summation * includes all the electrons of all the atoms in the molecule, while the origin of the second term will now be sought in the departure of the nuclear field from central

* It should be noted that the summation cannot be extended to all the electrons in the molecule, as some of them will be bi- or polynuclear. It is these electrons moving under the charges of more than one nucleus which contribute to the second term of (1). The way of looking at a molecule, *e. g.*, $CH_4$ as consisting of ions $C^{+4}$ and $H^-$, as has been done by some, is too artificial.

character. The nuclei act as distinct attracting centres, and so exert a torque upon the electrons, which causes a continual transference of angular momentum between the electrons and nuclei. Owing to this rapidly fluctuating character of electronic angular momentum the square of its average does not vanish even in ¹S states, though the average itself does.

Organic compounds have been found to obey the law of additivity to a first approximation, while inorganic compounds are not amenable to any such law. In the following we set ourselves to show that the diamagnetism of a molecule, organic or inorganic, may be expressed by a relation of the form

$$-\chi_{AB} = -\chi_A - \chi_B + \lambda,$$

where $\chi_{AB}$ is the susceptibility of the molecule, $\chi_A$'s the susceptibilities of the constituent atoms, and $\lambda$ a term which comprises various effects, whose nature may be determined from a study of the conditions which lead to molecule formation.

If $\psi_A$ and $\psi_B$ are two bond eigenfunctions, a necessary condition for the formation of a stable bond is that the resonance integral $J_x = \int \psi_A(1)\psi_B(2)\psi_A^*(2)\psi_B^*(1)d\tau$ corresponding to a permutation of the two AB bond electrons shall have as high a value as possible consistently with other conditions. This requires that $\psi_A$ and $\psi_B$ show maximum overlapping in the region between the two nuclei, leading thus to two effects. (1) Within the region of maximum overlapping the electrons are under the influence of both nuclei, and this causes the phenomenon of fluctuation of angular momentum as mentioned earlier. (2) A concentration of the eigenfunctions also means a concentration of the electric charge in that region, from which it follows that $\Sigma e^2 r^2 \psi \psi^* d\tau$ is reduced when the bond is formed.

To these two we can still add another. (3) In our picture of a molecule we imagine that every nucleus tends to be surrounded by an election density distribution corresponding to some stable electron configuration. This density distribution and the individual wave-functions have symmetries adapted to the configuration of the other nuclei surrounding the given nucleus. The unperturbed wave-functions of individual electrons split up into types adapted to the symmetry of the nuclear

configuration, and some of the new types may be such that the electrons whose behaviour they govern avoid the region of some of the nuclei. Such wave-functions reduce the energy of formation of the molecule, and their effect is to increase diamagnetism.

In general effect (2) is small, and in those cases where (3) is absent practically the whole of the loss in diamagnetism will be due to effect (1), which is measured by the magnitude of the resonance integral $J_\chi$ mentioned above. But for homopolar bonds this also measures the binding energy of the molecule. Hence we expect that in those cases where there is no " promotion " of electrons on molecule formation the loss in diamagnetism, meaning thereby the defect of the diamagnetic susceptibility of the molecule below the sum of the diamagnetic susceptibilities of the constituent atoms, will be proportional to the dissociation energy of the molecule. The greatest measure of agreement is expected for diatomic homopolar molecules.

For heteropolar molecules in which the constituent ions are separated from each other the change in diamagnetism on molecule formation may be calculated from the old and new quantum numbers of the transferred electron. If at equilibrium distances of the ions there is appreciable penetration of one ion into the electron cloud of the other, we shall get the phenomenon of fluctuation of angular momentum as mentioned earlier in the case of a homopolar bond. The relative magnitudes of the diamagnetic loss for homopolar and heteropolar compounds may thus be brought out. In purely homopolar bonds the binding energy is expressed by terms which denote Coulomb forces as also by the resonance terms, this latter being the more essential. In purely heteropolar bonds it is the electrostatic attraction of the ions which holds them together. The binding energy of a molecule is given by

$$E = E_r + E_e,$$

where $E_r$ is energy due to resonance and $E_e$ that due to Coulomb forces. For purely homopolar molecules $E_r \gg E_e$, while for purely heteropolar ones $E_r \ll E_e$. The loss in diamagnetism being proportional to $E_r$, this will be greater for homopolar molecules than for heteropolar ones.

The above conclusions regarding the relation between diamagnetic loss on molecule formation, type of the bond, and binding energy might be tested if we knew exactly the atomic values of the susceptibilities. Previous discussions have shown that we can calculate them approximately, and a comparison with experimental value reveals the fact that the best theoretical values are a little too low for elements of lower atomic number, while they are too high for higher values, agreement occurring in the neighbourhood of atomic number 18. Table V. brings out

## TABLE V.

### Diamagnetic Loss and Dissociation Energy.

| | Computed (atom+atom) (Slater). | Observed (molecule). | Difference between computed and observed values (per cent.). | Dissociation energy in volts. | Percentage loss per volt. |
|---|---|---|---|---|---|
| H$_2$ ... | 4·86 | 3·99 | 18 | 4·42 | 4 |
| C$_2$ .... | 18·82 | 11·76 | 37·5 | 7·0 | 5·5 |
| N$_2$ ... | 16·2 | 11·8 | 27·2 | 9·0(a) | 3·0 |
| Cl$_2$ ... | 40·78 | 41·5 | — | 2·54 | — |
| S$_2$..... | 44·02 | 30·72 | 30 | 4·9 | 6 |
| Br$_2$. .. | 67·57 | 62·4 | 7·7 | 1·96 | 4 |
| I$_2$ ..... | 102·86 | 91·5 | 11 | 1·53 | 7 |
| CN .... | 17·51 | 11·2 | 36 | 9·5 | 4 |

the relation between the diamagnetic loss and dissociation energy of some homopolar diatomic molecules.

(The experimental data are taken from the International Critical Tables except (a), which is from 'Nature,' cxxix. p. 870, 1932. Dissociation energy is obtained from spectroscopic data.)

The results for hydrogen may serve as a standard, and show a loss of 4 per cent. per volt of dissociation energy. Br$_2$ and CN give the same value. Carbon, sulphur, and iodine, all solids, give higher values, which seem to show that an allowance ought to be made for the heat of vaporization. Chlorine falls outside the scheme by giving an observed value which is higher than the computed one. The computed value corrected for binding energy is 36·7 for chlorine, and a redetermination will be of interest.

Going back to the question of interpenetration of ions, it may be added that while they are separated in very

dilute solutions there is always some penetration present in solids, concentrated solutions, and vapours. The diamagnetic susceptibility of solids or vapours of ionic compounds will, therefore, be lower than that calculated from dilute solutions. This is borne out for solids by Hocart's [11] measurements on NaCl, KCl, and $CaCl_2$ *.

### Table VI.

Diamagnetic Susceptibilities for Solids and Solutions.

|  | NaCl. | KCl. | $CaCl_2$. |
|---|---|---|---|
| Solid ............. | 30·1 | 39·1 | 54·45 (super-saturated) |
| Solution ........... | 30·8 | 39·6 | 55·6 |

As dissociation increases with dilution we shall also expect an increase in $\chi$ with increasing dilution. This is evident from the table below prepared from Hocart's measurements †.

### Table VII.

| | Concentration (per cent.) | $\chi_M$. | | Concentration (per cent.) | $\chi_M$. |
|---|---|---|---|---|---|
| | HCl. | | | KCl. | |
| I. | 7·35 | 22·16 | | 19·72 | 39·7 |
| | 12·96 | 22·05 | | 23·0 | 39·58 |
| | 14·3 | 21·67 | | | |
| II. | 15·68 | 22·08 | | | |
| | 26·85 | 21·26 | | | |

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* There is, of course, an interpenetration of the ions with the molecules of the solvent ; but the fact that the values for solutions are still higher than those for solids shows that the effect of this interpenetration is less than that of the ions of the same molecule.

† Farquharson's [16] measurements on HCl and $H_2SO_4$ show that the problem of variation of susceptibility with concentration is more complicated than usually pictured. The author hopes to discuss this in a future paper.

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In conclusion, I beg to offer my heartiest thanks to Prof. D. M. Bose for his helpful suggestions and valuable criticisms.

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LXIV. *The Diamagnetism of Free Electrons.*
*By* J. S. MITCHELL, *B.A.**

IT is now generally accepted that in the classical theory, the Maxwellian distribution law for the translational velocities of free charged particles is unaltered by a magnetic field, so that such a classical gas would have no diamagnetic (or paramagnetic) susceptibility†. However, it has been shown by Landau‡ that, according to quantum mechanics, a gas of free electrons has a small diamagnetic susceptibility due to the orbital motion about the field, in addition to the residual paramagnetism due to the intrinsic electron spin. The value of the diamagnetic susceptibility can be derived explicitly by a simple application of Fowler's statistical method §.

The assemblies of free electrons of most immediate interest are those in which the electron density $n_0$ is of the same order as the density of " free " electrons in metals, viz., $10^{22}$ per c.c. ; for such cases the partial potential $\lambda$, of the electrons in the assembly, is still great at 1000° K., so that the asymptotic expansion of the partition function integrals appears justified under ordinary experimental conditions.

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* Communicated by the Author.

† N. Bohr, Dissertation, Copenhagen, 1911; J. H. van Leeuwen, *J. de Phys.* (6) ii. p. 361 (1921).

‡ Landau, *Zeit. f. Physik*, lxiv. p. 629 (1930), see also Pauli, Report of Solvay Congress, 1930; Van Vleck, 'Electric and Magnetic Susceptibilities,' p. 353.

§ Fowler, 'Statistical Mechanics,' chap. xxi

The spectroscopic stability theorem shows that the suscepti-bility is invariant of the system of quantization, of the field strength, and of the manner of removal of any degeneracy. Magnetic distortion * is of no significance in first order approximations.

The wave-equation for a free electron moving in a magnetic field H, parallel to the $z$-axis, can be written

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m}{h^2}(\mathrm{E} - \mathrm{V})\psi$$
$$- \frac{8\pi^2 m}{h^2}\left(\frac{eh\mathrm{H}}{4\pi mc\iota}\frac{\partial \psi}{\partial \phi} + \frac{e^2\mathrm{H}^2 r^2}{8mc^2}\psi\right) = 0.$$

But $\mathrm{V} = 0$, and writing

$$\psi = e^{\iota n_1 \phi + \iota n_3 z} f(r),$$

$$\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \left\{\frac{8\pi^2 m}{h^2}\left(\mathrm{E} - \frac{n_3^2 h^2}{8\pi^2 m} - \frac{eh\mathrm{H}}{4\pi mc}n_1\right)\right.$$
$$\left. - \frac{n_1^2}{r^2} - \frac{\pi^2 e^2 \mathrm{H}^2}{h^2 c^2}r^2\right\} f = 0.$$

Putting

$$\nu_0 = \frac{e\mathrm{H}}{4\pi mc} \quad \text{and} \quad \mathrm{E}' = \mathrm{E} - \frac{n_3^2 h^2}{8\pi^2 m} - h\nu_0 n_1,$$

the proper functions are

$$\psi = \mathrm{C}e^{\iota n_1 \phi + \iota n_3 z}\frac{1}{r}\mathrm{W}_{\frac{\mathrm{E}'}{2h\nu_0},\,\frac{n_1}{2}}\left(\frac{4\pi^2 m\nu_0}{h}r^2\right),$$

where $\mathrm{W}_{k,m}(z)$ is the confluent hypergeometric function †, and the proper values are

$$\mathrm{E} = h\nu_0(2n_1 + 2n_2 + 1) + \frac{n_3^2 h^2}{8\pi^2 m}$$

$$= \mu_0 \mathrm{H}(2n + 1) + \frac{n_3^2 h^2}{8\pi^2 m},$$

where $n_1$, $n_2 = 0, 1, 2 \ldots$ and $n = n_1 + n_2 = 0, 1, 2, \ldots \mathrm{N}$ ; $\mu_0 = \frac{eh}{4\pi mc}$, the Bohr magneton.

The electron gas is enclosed in a large vessel of volume V, so that, at all ordinary temperatures and field strengths, the dimensions of the vessel are very great with respect to the

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* Niessen. *Zeit. f. Physik.* lviii. p. 63 (1929).
† See Whittaker and Watson, 'Modern Analysis,' chap xvi.

classical radius of curvature $r_1$ of the orbits ; but the weight of the element of phase space is of the same form $\dfrac{2e\mathrm{H}\mathrm{V}}{ch^2}\,dp_3$ when a cylindrical volume element of radius $r_1$ is considered. The boundary electrons are accounted for by taking N, the maximum value of $n$, to be the next integer less than $\dfrac{k\mathrm{T}}{2\mu_0\mathrm{H}}\log\lambda$. Thus the partition function for the assembly is

$$\mathrm{Z} = \sum_{n=0}^{\mathrm{N}} \frac{4e\mathrm{H}\mathrm{V}}{h^2 c}\int_0^\infty \log\left(1+\lambda e^{-\frac{\mu_0\mathrm{H}(2n+1)+\frac{p_3^2}{2m}}{k\mathrm{T}}}\right)dp_3,$$

and applying the theorem *

$$\sum_a^b f\left(x+\tfrac{1}{2}\right) = \int_a^b f(x)dx - \frac{1}{24}\left|f'(x)\right|_a^b,$$

assuming the asymptotic expansion

$$\int_0^\infty \log\left(1+e^{\beta-x}\right)x^{-1/2}dx \sim \frac{4}{3}\beta^{3/2} + \frac{\pi^2}{6}\beta^{-1/2} + \mathrm{O}[\beta^{-5/2}]$$

when $\beta$ is sufficiently large,

$$\mathrm{Z} \sim \sum_{n=0}^{\mathrm{N}} \frac{2e\mathrm{H}\mathrm{V}}{ch^2}(2mk\mathrm{T})^{1/2}\left\{\frac{4}{3}\left(\log\lambda - \frac{(2n+1)\mu_0\mathrm{H}}{k\mathrm{T}}\right)^{3/2}\right.$$
$$\left. + \frac{\pi^2}{6}\left(\log\lambda - \frac{(2n+1)\mu_0\mathrm{H}}{k\mathrm{T}}\right)^{-1/2}\right\}$$

$$= \frac{8e\mathrm{V}k\mathrm{T}}{15h^2c\mu_0}(2mk\mathrm{T})^{1/2}(\log\lambda)^{5/2}$$

$$- \frac{e\mu_0\mathrm{H}^2\mathrm{V}}{3h^2ck\mathrm{T}}(2mk\mathrm{T})^{1/2}(\log\lambda)^{1/2} + \frac{\pi^2 e\mathrm{V}k\mathrm{T}}{3h^2c\mu_0}(2mk\mathrm{T})^{1/2}(\log\lambda)^{1/2}.$$

The total number of systems †

$$\bar{\mathrm{M}} = \lambda\frac{\partial}{\partial\lambda}\mathrm{Z} = \frac{4e\mathrm{V}k\mathrm{T}}{3h^2c\mu_0}(2mk\mathrm{T})^{1/2}(\log\lambda)^{3/2} = n_0\mathrm{V},$$

* *Enzykl. der Math. Wiss.* ii. 2. p. 92 (quoted from Van Vleck).
† When

$$n_0 = 10^{22}\log\lambda = \left(\frac{3n_0}{\pi}\right)^{2/3}\frac{h^2}{8mk\mathrm{T}} = 1\cdot95\times\frac{10^4}{\mathrm{T}},$$

so that the asymptotic expansions are permissible under ordinary experimental conditions.

and the total resolved magnetic moment in the field direction

$$M = kT \frac{\partial}{\partial H} Z = -\frac{2e\mu_0 HV}{3h^2 c} (2mkT)^{1/2} (\log \lambda)^{1/2}$$

$$= -\frac{1}{3} \frac{4\pi m \mu_0^2}{h^2} \left(\frac{3n_0}{\pi}\right)^{1/3} HV$$

$$= KHV,$$

where K is the volume susceptibility, *i. e.*,

$$K = -\frac{1}{3} \frac{4\pi m \mu_0^2}{h^2} \left(\frac{3n_0}{\pi}\right)^{1/3}.$$

This is Landau's result, that, to a first approximation, the diamagnetic susceptibility of a gas of free electrons is one-third of Pauli's value for the paramagnetic susceptibility due to spin [*].

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LXV. *Emission from Oxide-coated Cathodes.* By M. BENJAMIN, *B.Sc., and* H. P. ROOKSBY, *B.Sc.*[†] (Communication from the Staff of the Research Laboratories of the General Electric Co., Ltd., Wembley, England, Jan. 1933.)

[Plates XVII. & XVIII.]

## INTRODUCTION.

IT is a matter of general experience that oxide-coated filaments when flashed at high temperatures, or when exposed to oxygen at ordinary running temperatures lose their emissions. Experience is greatest with mixed oxides, and it is found with these that flashing at sufficiently high temperatures or severe oxygen poisoning makes it impossible to recover completely the emissions by the processes normally employed in activating an inactive cathode.

In the case of flashed filaments, the permanent loss of emission has been ascribed by Lowry [‡] to crystal

[*] Pauli, *Zeit. f. Physik,* xli. p. 81 (1927) ; also Frenkel, *Zeit. f. Physik,* xlix. p. 31 (1928).

[†] Communicated by C. C. Paterson, M.I.E.E.

[‡] Lowry, Phys. Rev. xxxv. p. 1367 (1930).

growth.  Lowry's theory of electron diffusion through the coating is, however, now no longer tenable (Becker and Sears *).  Thus even if crystal growth does occur (and we have been unable to confirm this in the case of cathodes not exposed to air) it is difficult to see how it could affect the thermionic properties of the coating.

It was our view that an X-ray analysis of the coating in the activated, flashed, and poisoned states might yield some information about the composition and character of the coating.  Since, whether the emission is destroyed by flashing or poisoning, it is permanently lost, it seemed possible that the final condition of the coating might be the same in both cases.

Throughout we shall use the term "flashed filament" for a filament whose emission has been destroyed by high temperature flashing, and "poisoned filament" for one whose emission has been destroyed by exposure to oxygen.

### EXPERIMENTAL.

#### (a) *Description of Apparatus.*

The first essential was to devise a method for recording the X-ray diffraction pattern whilst the coated filament was in vacuum, as exposure of the filament to air would result in an alteration in the chemical nature of the coating.  The apparatus shown in fig. 1 was designed so as to permit removal of the filament after treatment to a thin-walled glass tube suitable for X-ray examination.

A nickel filament $F_1$ $F_2$, 20 mm. long from which the coating had been removed for 5 mm. at each end, was suspended from the seal by a flat nickel spring $S_1$.  The filament was surrounded about its coated portion by a cylindrical nickel anode 10 mm. in diameter and 5 mm. long.  Bearing on opposite sides of the uncoated portions of the filament, and about 2 mm. from each end, were two short lengths of 0·15 mm. molybdenum wire $M_1$, $M_2$ hooked at the ends.  A flat nickel disk G on to which was clipped a piece of magnesium for gettering purposes was placed parallel to the seal, the magnesium facing outwards.

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* Becker and Sears, Phys. Rev. xxxviii. p. 2193 (1931).

The whole was sealed into a glass envelope which could be evacuated by means of an exhaust tube from the seal. Attached to the top of the envelope was a very thin-walled capillary tube $T_1$, made of soda glass. This tube was totally enclosed by an outer tube $T_2$, which could be

Fig. 1.
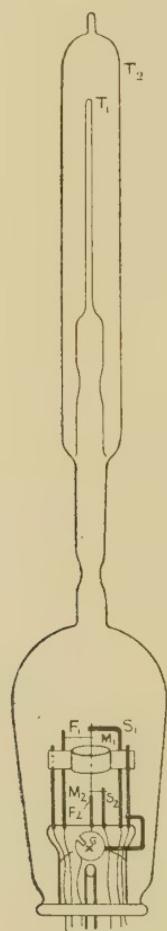


Tube for preparation of specimens for X-ray examination.

evacuated. It was found necessary to evacuate $T_2$ in order to prevent collapse of the capillary during baking.

The filament after treatment was detached by passing a current round $M_1F_1S_1$ and $M_2F_2S_2$ in turn, until the filament just burnt out at $M_1$ and $M_2$. With the tube in a vertical position, the capillary pointing downwards,

the filament fell straight into the capillary. The outer tube $T_2$ was removed and the capillary sealed off.

The X-ray examination was made by the Debye-Scherrer method. $CuK\alpha$ radiation after passing through a suitable pinhole system impinged upon the filament in the capillary tube. The latter was mounted on the axis of a cylindrical camera, of 6 cm. radius, which held a strip of photographic film.

## (b) *Preparation of the Filaments.*

The core metal used was nickel (99·5 per cent.+Ni by chemical analysis) in the form of a wire 0·044 mm. diameter. This wire, after cleaning and annealing in hydrogen at 700° C., was coated continuously by passing it through a series of baths containing a suspension of strontium and barium carbonates in a suitable liquid medium. The proportions were such that subsequent reduction to oxides assuming a simple decomposition, should yield 54 per cent. SrO—46 per cent. BaO by weight. After each application of the mixture, the coating was sintered on to the wire by passing through an oven at 700° C. in an atmosphere of $CO_2$. The coating weight was in the neighbourhood of 0·25–0·35 mg. per 20 cm. length of wire.

## (c) *Preparation of Specimens.*

The procedure up to the activation stage was the same for each tube. The filaments were assembled in the apparatus as described. The anode and getter disk were out-gassed before assembly by vacuum-cooking at 1000° C. The magnesium was similarly treated, but at 500° C.

Pumping was carried out on a three-stage mercury vapour diffusion pump employing a liquid-air trap between pump and apparatus. The outer tube $T_2$ having been evacuated, the tube itself was evacuated and baked at 400° C. for 30 minutes. The anode was then further out-gassed by high-frequency induction heating. The filament was lit at a temperature of 1250° K.* for 30 seconds in order to decompose the carbonates to oxides ; and the magnesium dispersed by high-frequency induction heating. The tube was then sealed off.

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\* All filament temperatures given in the text are approximate only.

(d) (1) *Activated Filament.*

The filaments were activated by drawing space current from them whilst running at 1250° K. for a 9 minute period, and at 1150° K. for 6 minutes. Emission measurements were made at 1020° K.

After activation the filaments were detached and the X-ray examination made.

(2) *Flashed Filament.*

After activation the filaments were flashed for one minute intervals at 1600° K.

After each flash the filament was reactivated. This process of alternate flashing and reactivation was continued until no further reactivation was possible. This state was deemed to have occurred when the final space current obtained after reactivation was only a very small fraction of the original space current. The final emission of the filament at the measuring temperature was then only of the order of 0·2 milli-amps. compared with its original value of 7 milli-amps. The filament was then detached for the X-ray examination.

## X-Ray Results.

(a) *Nature of Coating.*

A preliminary investigation was made of the nature of the coating on the wire, prepared as described above before introduction into the valve. X-ray examination revealed that the coating was composed of a solid solution of barium and strontium carbonates, one in the other, and a typical X-ray pattern is shown in fig. 2 (*a*) (Pl. XVII.)

The distinction between a mixture and a solid solution of two compounds must be clearly understood, and, since recognition of the difference is essential to the main argument that follows, the case of the carbonates will be briefly discussed here. Barium and strontium carbonates having isomorphous crystals give very similar X-ray patterns, but owing to the difference in size between the barium and strontium atoms, the lines are more spread out for strontium carbonate. The lines in the pattern of the coating are intermediate in spacing, thus showing it to consist of a solid solution of the two carbonates. A mixture would have given the two individual

X-ray patterns superimposed. The position of the lines, with respect to the positions of the corresponding ones in the individual patterns of barium and strontium carbonates can be used to estimate approximately the composition of the solid solution, a linear law being assumed to connect the change in atomic spacings with change in molecular composition. The composition obtained in this way for the coating was 41 per cent. $BaCO_3$, 59 per cent. $SrCO_3$, corresponding to 43 per cent. BaO, 57 per cent. SrO by weight, or in molecular proportions 34 per cent. BaO, 66 per cent. SrO. These values are in good agreement with those calculated assuming a simple decomposition of the original coating mixture, and are within the experimental error of measurement.

An account of the results of the examination of several activated and flashed filaments will now be given, the compositions of the solid solutions formed of barium and strontium oxides, revealed by the examination, having been estimated from measurement of the X-ray patterns in a similar manner to that discussed for the carbonates. Further reference to the relation between atomic spacings and molecular composition will be made in a subsequent section.

Photographs of activated filaments were taken by the method described above and a typical one is reproduced in fig. 2(c) (Pl. XVII.). The broken lines in the pattern are from the nickel core, which contained comparatively large crystals.

As standards of comparison, X-ray patterns of two filaments coated with barium oxide and strontium oxide respectively were obtained, and these are shown in fig. 2 (b) and (d) (Pl. XVII.). Barium and strontium oxides are of face-centred cubic crystal form, and it will be observed that the lines due to the coating, fig. 2 (c) (Pl. XVII.) are due to a face-centred lattice of intermediate spacings. That is, the coating is composed of a solid solution of barium and strontium oxides, one in the other. Measurement of the spacing of the lines gave an approximate value for the composition of the coating as 40 per cent. BaO, 60 per cent. SrO by weight, or in molecular proportions 30 per cent. BaO, 70 per cent. SrO. The lengths of the sides of the unit cubes of barium and strontium oxides used in this calculation were 5·515 A.U.

and 5·135 A.U. respectively, these values having been
determined from the above-mentioned photographs,
fig. 2 (*b*) and (*d*) (Pl. XVII.).    Our values differ slightly
from those given by W. Gerlach *, but are possibly nearer
the true values, as they have been determined with refer-
ence to the lines on the same photographs for pure nickel,
whose atomic spacings are accurately known †.

A theory ‡ has recently been advanced, in a discussion
of the mechanism of emission from oxide-coated filaments,
that, in the activated condition, part of the coating was
present as a suboxide of barium.    The X-ray photographs
we have obtained show that the bulk of the coating is
present as monoxide, and give no indication of the
existence of any other oxide in the coating.    The main
portion of the latter is therefore composed of monoxide,
but the X-ray method is unsuitable for the detection
of small proportions of additional materials, and hence
it is possible that a small quantity of suboxide could exist.

The next step was to examine filaments whose emission
had been destroyed by flashing at high temperature.    The
X-ray patterns (a typical one is reproduced in fig. 2 (*e*)
(Pl. XVII.) showed that the coating consisted of a solid
solution of barium and strontium oxides, but comparison
with the photographs of the activated filaments revealed
that the composition was different.    Calculations from
the spacing of the lines revealed an average composition
for the solid solution of 91 per cent. SrO, 9 per cent. BaO
by weight, or in molecular proportions 94 per cent. SrO,
6 per cent. BaO.    This means that a large part of the
barium oxide had been entirely lost from the coating.
We have confirmed this disappearance of barium oxide
in the case of a filament whose emission had failed after
running for several hundred hours at 1100° K.    It thus
appears evident that high temperature treatment results
in preferential loss of barium oxide.    This point will be
amplified further in a subsequent paragraph.

It is impossible from the X-ray photographs we have
obtained to gather any definite information as to crystal
size.    A tentative conclusion that the upper limit is
$10^{-3}$ cm. may be drawn from the fact that in the X-ray

* W. Gerlach, *Zeits. f. physik*, ix. p. 184 (1922).
† L. Mazza & A. G. Nasini, Phil. Mag. vii. p. 301 (1929).
‡ Meyer & Schmidt, *Z. Tech. Phys.* iii. p. 137 (1932).

patterns the lines due to the coating are unbroken and continuous. Again, the sharpness of the lines would lead us to suppose that the lower limit of crystal size is about $10^{-5}$ cm. Finally, we can state that we detect no difference by this method between the crystal size in an inactive coating and that in a flashed coating, but the limits are obviously wide enough to allow of considerable crystal growth having taken place.

Microscopic examination of activated and flashed filaments, whilst still in the glass capillary tubes, failed to reveal any tangible difference in the appearance of the coatings. Typical photographs are reproduced in fig. 3 (Pl. XVIII.). We were thus unable to confirm the results obtained by Lowry, although it is possible that differences in the method of preparation of the filaments may be responsible. We mention this, as our filament coatings after activation are not of a fluffy appearance.

### (b) *Discussion.*

The results described in the preceding section enable us to offer an explanation for the loss of emission due to high temperature flashing. The loss of emission is associated with a loss of barium oxide * from the coating. J. W. Ryde and D. E. Yates †, of these Laboratories, have shown spectroscopically that the failure of emission during life, in the case of BaO filaments prepared by the " barium vapour " process is accompanied by a gradual loss of barium oxide.

We have established, however, that the " paste " form of barium/strontium oxide filament undergoes a preferential loss of barium oxide. The consequent change in composition of the coating, increasing the proportion of strontium oxide would, as we shall show later, be expected to result in a much reduced emission.

### Loss of Coating by Weight.

The loss of coating, as revealed by the X-ray analysis, was great enough to be detectable by actual weighings of activated and flashed filaments.

A number of short sections of the coated wire, 68 mm. in length, were weighed on a torsion balance which read

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* Whether as Ba or BaO we have not yet determined.
† Unpublished work.

accurately to 0·001 milligram. The coatings were re-
moved from the cores by washing in dilute acetic acid.
The cores were then washed, dried, and weighed, and
thus the mean weight of coating and core before treatment
estimated.

A number of similar filaments were then assembled
such that each filament was surrounded by a nickel
anode, and the whole sealed into a glass envelope. The
filaments were activated in the manner already described.
After activation some of the filaments had their emissions
destroyed by flashing. The activated and flashed fila-
ments were removed from the tubes and the coatings
converted back into carbonates by heating in a $CO_2$
atmosphere at 700° C. until the filament weights were
constant. The centre portions of the filaments, about
10 mm. in length, were then weighed and weights of
coating and core determined. The results are given
In Table I. The final composition of the coating as
oxides is calculated assuming that only BaO is lost.

## TABLE I.

### Data showing Change in Composition of Coating after Activation and Flashing.

(a) *After Coating.*

| Wt. of core + coating per 10 mm. length. | Wt. of core per 10 mm. length. | Wt. of coating per 10 mm. length. | Composition as oxides. | |
|---|---|---|---|---|
| | | | BaO. | SrO. |
| mg. | mg. | mg. | | |
| $13·20 \times 10^{-2}$ | $11·95 \times 10^{-2}$ | $1·25 \times 10^{-2}$ | | |
| 13·09    ,, | 11·86    ,, | 1·23    ,, | | |
| 13·09    ,, | 11·85    ,, | 1·24    ,, | | |
| 13·10    ,, | 11·90    ,, | 1·20    ,, | | |
| 13·09    ,, | 11·90    ,, | 1·19    ,, | 43·0 p. c. | 57·0 p. c. |
| 13·10    ,, | 11·85    ,, | 1·25    ,, | | |
| 13·09    ,, | 11·86    ,, | 1·23    ,, | | |
| 12·99    ,, | 11·80    ,, | 1·19    ,, | | |
| 13·10    ,, | 11·86    ,, | 1·24    ,, | | |
| Mean 13·10    ,, ±1 per cent. | Mean 11·86    ,, ±1 per cent. | Mean 1·22    ,, ±3 per cent. | | |

## (b) *After Activation.*

| Wt. of core +coating. | Wt. of core. | Wt. of coating. | Loss of core. | Loss of coating. | Composition as oxides. | |
|---|---|---|---|---|---|---|
| | | | | | BaO. | SrO. |
| mg. | mg. | mg. | per cent. | per cent. | per cent. | per cent. |
| $12 \cdot 85 \times 10^{-2}$ | $11 \cdot 62 \times 10^{-2}$ | $1 \cdot 23 \times 10^{-2}$ | 2 | 0·82 | 43 | 57 |
| 12·90 ,, | 11·80 ,, | 1·10 ,, | 0·5 | 9·85 | 37 | 63 |
| 13·00 ,, | 11·86 ,, | 1·14 ,, | 0 | 6·55 | 40 | 60 |
| 13·05 ,, | 12·00 ,, | 1·05 ,, | 1·2 | 13·90 | 33 | 67 |
| | | | | | Mean 38 | 62 |

## (c) *After Flashing.*

| Wt. of core + coating. | Wt. of core. | Wt. of coating. | Loss of core. | Loss of coating. | Composition as oxides. | |
|---|---|---|---|---|---|---|
| | | | | | BaO. | SrO. |
| mg. | mg. | mg. | per cent. | per cent. | per cent. | per cent. |
| $12 \cdot 60 \times 10^{-2}$ | $11 \cdot 86 \times 10^{-2}$ | $0 \cdot 74 \times 10^{-2}$ | 0 | 39·4 | 3 | 97 |
| 12·20 ,, | 11·50 ,, | 0·70 ,, | 3·0 | 42·5 | 0 | 100 |
| 12·35 ,, | 11·60 ,, | 0·75 ,, | 2·2 | 38·5 | 4·5 | 95 |
| 12·10 ,, | 11·50 ,, | 0·60 ,, | 3·0 | 50·6 | 0 | 100 |
| 12·25 ,, | 11·50 ,, | 0·75 ,, | 3·0 | 38·5 | 4·5 | 95·5 |
| 12·02 ,, | 11·45 ,, | 0·57 ,, | 3·5 | 53·0 | 0 | 100 |
| 12·42 ,, | 11·65 ,, | 0·82 ,, | 2·2 | 32·8 | 13 | 87 |
| 12·25 ,, | 11·45 ,, | 0·80 ,, | 3·5 | 34·4 | 11 | 89 |
| | | | | | Mean 5 | 95 |

This assumption is justified by comparison with the X-ray results and by a subsequent experiment with strontium oxide which shows a relatively small loss of coating on flashing. It will be seen from the table that the order of the results agrees with that obtained from the X-ray photographs.

### DEPENDENCE OF EMISSION ON COMPOSITION.

We have shown that the loss of emission on flashing can be associated with a preferential loss of BaO from the coating. This loss involves a change in the composition of the coating, increasing considerably the proportion

of SrO. It is well known that for " paste " oxide filaments there is an optimum value of composition which is obtained by coating the filament with a mixture in the neighbourhood of 50 per cent. BaO—50 per cent. SrO by weight. Any loss of BaO involves a shift in composition, and as the proportion of SrO increases so the emission will tend to decrease. From our results it follows that, starting with a coating rich in BaO, flashing should result in an increasing emission as the optimum is approached, and a falling emission as it is passed.

A wire was coated with a mixture designed to give 80 per cent. BaO—20 per cent. SrO by weight on decomposition of the carbonates to oxides. 68 mm. lengths of this wire were mounted up with anodes, sealed into glass envelopes, and activated under conditions already described. To allow for any differences in the thermal emissivity of the coating as its composition changed, measurements and reactivation were made with the hot resistances of the filaments kept constant throughout any particular set of conditions. This method was adopted whenever filaments with different coatings were used. The filaments were then flashed at 1300° K. for 3 minute intervals and reactivated after each flash. This lower flashing temperature was used in order to observe the emission changes in greater detail. The change of emission with flashing for a typical filament is shown in fig. 4. Positions 20 and 21 indicate flashing at 1500° K. in order to hasten the end.

Referring to the curve it will be seen that during the flashing the emission rises, reaches a maximum, and then begins to fall steadily. It will be observed that the maximum is very flat, indicating that there is a fairly wide range of composition, over which the greatest emission is obtainable. For purposes of comparison, we show a curve (fig. 4 *a*) which we have obtained connecting emission with coating composition of the activated filament. It will be seen that as the strontium end of the curve is approached, the emission begins to fall rapidly, and agreement is of the right order to explain the loss of emission as being due to change in composition of the oxides.

The full curve shows the compositions, as put on the filaments before any heat treatment. The broken curve shows the compositions after activation as determined

by X-ray analysis, the corresponding molecular pro-
portions being also given. It will be observed that the
correspondence between the original composition and that

Fig. 4.
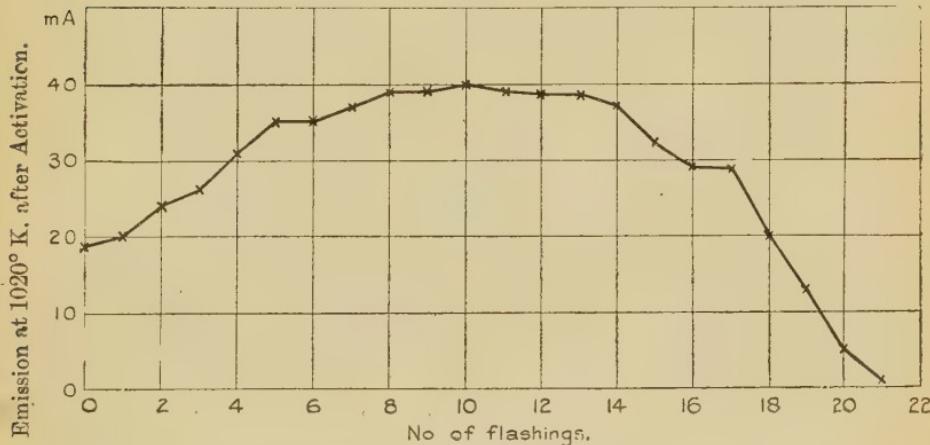


Change of emission of 80 per cent. BaO–20 per cent. SrO coated
filament with repeated flashing and reactivation.

Fig. 4 *a*.
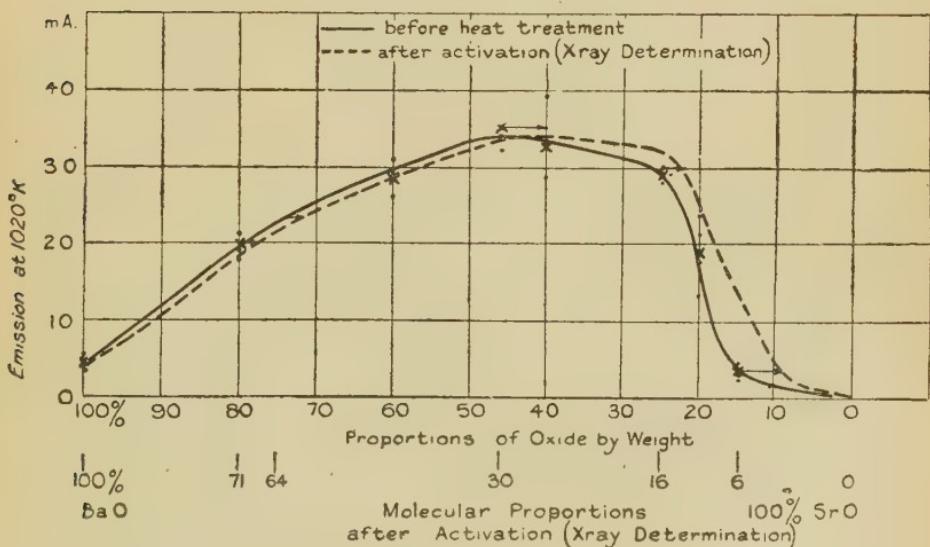


determined by the X-ray method is in each case within
a few per cent. There is, indeed, a slight deficiency
in barium oxide content, but this deficiency might be
expected on consideration of the previous results on high

temperature flashing. The fact that the correspondence is so good, however, justifies the assumption of the

## TABLE II.

Data showing Change of Composition of 80 per cent. BaO — 20 per cent. SrO Filament after Flashing.

(a) *Before Treatment.*

| Wt. of core+coating. | | Wt. of core. | | Wt. of coating. | |
|---|---|---|---|---|---|
| 13·30 mg. $\times 10^{-2}$ | | 11·90 mg. $\times 10^{-2}$ | | 1·40 mg. $\times 10^{-2}$ | |
| 13·38 | ,, ,, | 11·84 | ,, ,, | 1·54 | ,, ,, |
| 13·38 | ,, ,, | 11·95 | ,, ,, | 1·43 | ,, ,, |
| 13·40 | ,, ,, | 11·85 | ,, ,, | 1·56 | ,, ,, |
| 13·38 | ,, ,, | 11·88 | ,, ,, | 1·50 | ,, ,, |
| 13·30 | ,, ,, | 11·85 | ,, ,, | 1·55 | ,, ,, |
| Mean.... 13·34 | ,, ,, | 11·85 | ,, ,, | 1·49 | ,, ,, |

(b) *After Complete Flashing.*

| Wt. of core +coating. | Wt. of core. | Wt. of coating. | Loss in wt. | Composition as oxides. | |
|---|---|---|---|---|---|
| | | | | BaO. | SrO. |
| mg. | mg. | mg. | per cent. | per cent. | per cent. |
| $12·06 \times 10^{-2}$ | $11·70 \times 10^{-2}$ | $0·36 \times 10^{-2}$ | 76·0 | 12 | 88 |
| 12·02 × ,, | 11·70 × ,, | 0·32 × ,, | 78·5 | 0 | 100 |
| | | | | Mean.... 6 | 94 |

(c) *After Flashing to Optimum.*

| Wt. of core +coating. | Wt. of core. | Wt. of coating. | Loss in wt. | Composition as oxides. | |
|---|---|---|---|---|---|
| | | | | BaO. | SrO. |
| mg. | mg. | mg. | per cent. | per cent. | per cent. |
| $12·32 \times 10^{-2}$ | $11·68 \times 10^{-2}$ | $0·64 \times 10^{-2}$ | 57·0 | 45 | 55 |
| 12·25 ,, | 11·72 ,, | 0·53 ,, | 64·5 | 41 | 59 |
| | | | | Mean.... 43 | 57 |

linear relation between molecular composition and atomic spacing in the X-ray calculations.

Table II. gives some weighing results on the 80 per cent. BaO filaments after complete flashing, and after flashing

to the centre of the flat portion of the curve in fig. 4. It will be observed that complete flashing results in a considerable loss of BaO from the coating. The results agree in order of magnitude with those obtained in Table I., and would indicate an optimum value of composition in the region of 43 per cent. BaO—57 per cent. SrO by weight or 34 per cent. BaO, 66 per cent. SrO in molecular proportions.

## FLASHING OF SINGLE OXIDE COATINGS.

It follows from our results that commencing with coatings of single oxides, it should be possible to recover the emission after each flash, so long as any coating remains. Wires were coated with pure BaO and pure SrO and diode tubes made as previously described. The filaments, together with a mixed oxide coated filament, were activated as before, then flashed and reactivated in a series of steps as follows :—

(1) 1350° K. for 1 minute.
(2) 1430° K. for 1 minute.
(3) 1430° K. for 5 minutes.
(4) 1520° K. for 1 minute.
(5) 1520° K. for 5 minutes.
(6) 1600° K. for 1 minute.
(7) 1600° K. for 5 minutes.

It was observed that, whilst there was no difference between the thermal emissivities of the mixed oxide filament and the pure SrO filament, that of the pure BaO filament was lower, and that during flashing this still further decreased.

There was no sensible change in the thermal emissivities of the other two filaments during flashing.

Referring to fig. 5, it will be seen that the emission of BaO remains constant and that of SrO tends to rise * somewhat. That of the mixed oxide filament falls steadily. Table III. gives the results of weight measurements on the pure SrO and pure BaO filaments.

It will be seen that the loss of weight of SrO from the SrO surface is small compared with the loss of weight of BaO from the BaO surface. We have since learnt

* This may possibly be due to an ageing effect.

that A. Gehrts * has found spectroscopically that during evaporation of active $BaO-SrO$ coatings the Ba line is very much stronger than the Sr line in a discharge produced in the inter-electrode space.

Fig. 5.
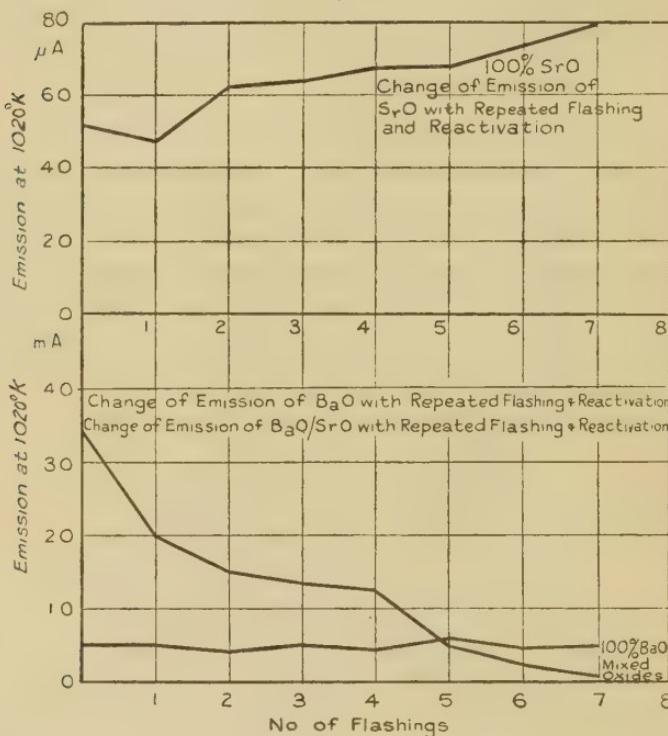


TABLE III.

Data showing Loss of Coating of SrO and BaO Filaments after Flashing.

SrO.

(a) *Before Treatment.*

| Wt. of core+coating. | | | Wt. of core. | | | Wt. of coating. | | |
|---|---|---|---|---|---|---|---|---|
| 12·90 mg. $\times 10^{-2}$ | | | 11·80 mg. $\times 10^{-2}$ | | | 1·10 mg. $\times 10^{-2}$ | | |
| 12·81 | ,, | ,, | 11·76 | ,, | ,, | 1·05 | ,, | ,, |
| 12·95 | ,, | ,, | 11·90 | ,, | ,, | 1·05 | ,, | ,, |
| 12·72 | ,, | ,, | 11·76 | ,, | ,, | 0·96 | ,, | ,, |
| Mean..12·85 | ,, | ,, | 11·80 | ,, | ,, | 1·05 | ,, | ,, |

* *Naturwissenschaften,* xx. p. 732 (1932).

## (b) *After Flashing.*

| Wt. of core + coating. | Wt. of core. | Wt. of coating. | Loss of coating. |
|---|---|---|---|
| mg | mg. | mg. | per cent. |
| $12{\cdot}65 \times 10^{-2}$ | $11{\cdot}65 \times 10^{-2}$ | $1{\cdot}00 \times 10^{-2}$ | 5 |
| $12{\cdot}80$ ,, | $11{\cdot}76$ ,, | $1{\cdot}04$ ,, | 1 |

### BaO.

## (a) *Before Treatment.*

| Wt. of core + coating. | | Wt. of core. | | Wt. of coating. | |
|---|---|---|---|---|---|
| $12{\cdot}65$ mg. $\times 10^{-2}$ | | $11{\cdot}76$ mg. $\times 10^{-2}$ | | $0{\cdot}89$ mg. $\times 10^{-2}$ | |
| $12{\cdot}90$ ,, | ,, | $11{\cdot}86$ ,, | ,, | $1{\cdot}04$ ,, | ,, |
| $12{\cdot}84$ ,, | ,, | $11{\cdot}80$ ,, | ,, | $1{\cdot}04$ ,, | ,, |
| $12{\cdot}80$ ,, | ,, | $11{\cdot}80$ .. | ,. | $1{\cdot}00$ ,, | ,, |
| $12{\cdot}83$ ,, | ,, | $11{\cdot}80$ ,, | ,, | $1{\cdot}03$ ,, | ,, |
| Mean..$12{\cdot}81$ ,, | ,, | $11{\cdot}80$ ,, | ,, | $1{\cdot}00$ ,, | .. |

## (b) *After Flashing.*

| Wt of core + coating. | Wt. of core. | Wt. of coating. | Loss of coating. |
|---|---|---|---|
| mg. | mg. | mg. | per cent. |
| $12{\cdot}28 \times 10^{-2}$ | $11{\cdot}70 \times 10^{-2}$ | $0{\cdot}58 \times 10^{-2}$ | 42 |
| $12{\cdot}25$ ,, | $11{\cdot}65$ ,, | $0{\cdot}60$ ,, | 40 |

### Oxygen Poisoning of a Mixed Oxide-coated Filament.

An X-ray examination was made of a filament similar to those already described which had been poisoned by exposure to oxygen for a short time. The apparatus, which contained two getter disks, was provided with a side-tube containing a few crystals of potassium permanganate. Activation was carried out as before, but on this occasion with the apparatus still connected to the pump. After activation, oxygen at a pressure of 0·01 mm. Hg. was admitted to contact with the filament, which was maintained at its normal operating temperature. The emission immediately fell to zero. After a few minutes interval, the oxygen was pumped away and the second getter disk dispersed. On now

attempting to reactivate the filament in the usual manner, it was found impossible to obtain a space current of more than 300 $\mu$a. as compared with the original space current of 20 ma. at the same temperature. This space current showed no sensible increase over the whole period of activation, and the final emission was only 25 $\mu$a. compared with the original emission of 7 ma. at the measuring temperature. The filament was now detached as before for X-ray analysis. The X-ray photograph revealed no sensible difference in coating composition from that of the activated filament.

Thus the condition brought about by oxygen poisoning is different from that resulting from flashing, and since the poisoned filament cannot be reactivated in the normal manner, its condition must be different from that of a filament which has not been activated for the first time, and whose carbonates have just been broken down to oxides. The oxygen poisoning must consist of more than a mere oxidation of the uncombined barium in the coating. In addition, it must result in some surface condition (adsorbed oxygen ?) which not only renders the filament thermionically inactive, but is also stable at the normal activating temperature. At sufficiently high temperatures surface oxygen must, of course, evaporate. But it has been shown that if the activating temperature is much exceeded, serious evaporation of barium oxide occurs and the potential activity of the filament reduced on this account. The next experiments were therefore made with filaments coated with single oxides, partial evaporation of which cannot affect the nature of the residual coating.

### POISONING AND RECOVERY OF FILAMENTS COATED WITH SINGLE OXIDES.

Experiments similar to the above were made with filaments coated with single oxides. The filaments were 68 mm. long and were mounted in a manner similar to that already described for filaments of this length. Qualitatively the same phenomena were observed. It was now found, however, that by raising a filament to near the melting-point of nickel, a rapidly increasing space current could be obtained, brought about, presumably, by evaporation of surface oxygen. And on reducing

the temperature to the normal value for activation, the space current continued to grow until, eventually, the filament was completely reactivated. Table IV. shows how the emission of two filaments, one coated with BaO and the other coated with SrO, was recovered after three successive poisonings. It will be observed that the recovery of the SrO coated filament was actually more than complete on each occasion, successively higher emissions having been obtained. This was possibly an ageing effect.

## TABLE IV.

Data showing Changes in Emission after Poisoning and Reactivation of Single Oxide Filaments.

|  | Original emission. | Emission after poisoning. | Emission after reactivation. |
|---|---|---|---|
| BaO coated filament. | 5·2 μa. | (1) 0·2 μa.<br>(2) 0 ,,<br>(3) 0 ,, | 5·0 ma.<br>4·8 ,,<br>5·0 ,, |
| SrO coated filament. | 80 μa. | (1) 0·06 μa.<br>(2) 0·01 ,,<br>(3) 0·08 ,, | 200 μa.<br>220 ,,<br>260 ,, |

### RECOVERY OF EMISSION BY SPUTTERING.

It would appear from these experiments that if only the surface layer inhibiting the emission of a poisoned BaO—SrO coated filament could be removed otherwise than by flashing, the emission of such a filament should also be recoverable by reactivating. Now a well-known method of removing surface atoms is that of sputtering. For sputtering to be effective, it is necessary for the sputtering ion to be massive. Since both argon and mercury fulfil this condition, can be obtained very pure, and can have no chemical effect on the oxide cathode, it was decided to try these.

After the filament had been activated and poisoned on the pump, excess oxygen was pumped away, getter dispersed from the second disk, and an unsuccessful attempt made to reactivate the filament in the normal manner. Spectroscopically pure argon was then admitted

to a pressure of about 0·01 mm. Hg, and a glow discharge started with a high-frequency induction coil and maintained by electrons derived from the filament heated to its activating temperature. The space current was limited to 120 ma. by means of a resistance in series with the anode, the voltage of the latter varying between 15 and 25 volts.

After 5 minutes bombardment, the argon was pumped away and the filament reactivated in the normal manner for 15 minutes. It was observed that the space current developed almost immediately and that at the end of the period the emission of the filament was fully recovered. This process was repeated a number of times on the same filament after successive poisonings.

Similar experiments with mercury vapour were only qualitatively successful. This is, we believe, attributable to the fact that in order to obtain a reasonable pressure of mercury vapour, the whole apparatus must be warmed up ; and that after the bombardment is complete, it is necessary to bake the apparatus in order to drive off mercury from the electrodes and glass work. This probably gives rise to fresh poisoning. Prolonged running of the filament at 1020° K. whilst taking a small space current from it resulted in the emission increasing to the right order.

It is interesting to note that L. R. Koller*, among others, has recorded the beneficial effects of positive ion bombardment, using gases which include argon. He states that his emissions were increased up to a hundredfold by such means, and that they were maintained when the gases were removed. From the relatively low order of emissions usually obtained at that time from oxide cathodes, however, it would appear that they were nearly always in a partially poisoned condition, and that the effect of the positive ion bombardment was merely to bring about recovery from this poisoning.

## Summary.

(1) It has been shown by X-ray analysis that the only material present in bulk form in the coating of an active cathode is the monoxide. If the coating consists of a mixture of barium and strontium oxides, these form a solid solution in one another.

* L. R. Koller, Phys. Rev. xxv. p. 671 (1925).

(2) If the coating is of one oxide only and the emission is destroyed by flashing, the remaining coating may be fully reactivated by repeating the activating process.

(3) Recovery of the original activity is not possible after severe flashing of a mixed BaO/SrO coated cathode. This has been shown to be associated with a preferential loss by evaporation of BaO from the coating, and is thus attributable to an altered composition.

(4) If the emission of a cathode coated with a single oxide has been poisoned by oxygen, it may always be completely recovered by first flashing at sufficiently high temperature with space current, and then re-activating.

(5) In the case of a mixed BaO/SrO coated cathode it is not always possible to recover the emission in this way, presumably because surface oxygen cannot be "flashed off" without serious evaporation of BaO occurring at the same time.

(6) Recovery, however, is possible in this case also, if, instead of flashing, the surface layer is sputtered off by ions of argon or mercury.

LXVI. *Electrodeless Discharges in Argon.*—Part I. *Electrical Properties. By* S. P. McCallum, *M.A., Fellow of New College, University Demonstrator, Oxford; and* L. Klatzow, *B.Sc., Lincoln College, Oxford* *.

MANY investigations have been made of the potentials required to maintain discharges in gases with high-frequency oscillations, and also of the potentials necessary to start the discharges. While the general proper ies of these discharges are similar in monatomic and in diatomic gases, there is a remarkable difference between the potentials required to maintain the discharges in the

two types of gas. Experiments have been made in the Electrical Laboratory, Oxford, to determine the electric force Z in the long, uniform, luminous columns in such discharges. Discharges can be maintained in monatomic gases at high pressures in wide tubes when the ratio of the force Z in the uniform positive column, to the pressure $p$, is comparatively small. Thus, in tubes about 3 cm. in diameter the force in the positive column of a discharge in neon at 20 mm. pressure is about 9 volts per cm. $(Z/p=0.45)$. In tubes of the same diameter with helium at the same pressure, the force is about 30 volts per cm. $(Z/p=1.5)$. In nitrogen, however, at 5 mm. pressure the force is about 100 volts per cm. $(Z/p=20)$.

It has recently been found possible to prepare argon free from any impurities that could be detected by spectroscopic methods, or by measurements of the electrical properties of the gas. The high-frequency discharges in argon are in many respects similar to those observed in other gases, but there are some marked differences in the phenomena observed in argon from those in helium and neon, which we have investigated.

It has been shown by Johnson[*] and by Jones[†] that the forces in the luminous positive columns of the direct current discharge in neon and in helium are the same as the forces in the luminous columns of the high-frequency discharge, and it has been shown by Townsend[‡], from theoretical considerations, that the force in the two types of discharge should be the same under the same conditions of pressure and diameter of tube.

Owing to difficulties in the purification of argon, it was thought advisable to investigate the properties of the luminous positive column of this gas by using high-frequency oscillations, in order to avoid impurities which may be emitted from electrodes in the gas.

In order to obtain consistent results with argon it is necessary to purify the gas very carefully. It was found that impurities such as hydrogen and nitrogen have a larger effect on the conductivity of argon than on that of helium or neon. For example, at a pressure of about 2 mm., the introduction of about 0.1 per cent. of nitrogen to pure argon raised the value of the force in the positive

[*] P. Johnson, Phil. Mag. x. p. 921 (1930).
[†] F. Ll. Jones, Phil. Mag. xi. p. 163 (1931).
[‡] J. S. Townsend, Phil. Mag. xi. p. 1113 (1931); xiii. p. 745 (1932).

column of a discharge from 2 volts per cm. to 9 volts per cm. It is to be expected that the effects of these gases would be exceptionally large in argon, because, as has been shown *, when small quantities of hydrogen are added to argon, the mean energy of the electrons in the gas is reduced by a large amount. It would therefore be necessary to increase the value of $Z/p$ considerably in order to maintain the mean energy of the electrons in the mixture at a value sufficiently great to produce ionization by collision.

The experiment shows that the effects of direct collisions of electrons with molecules of impurities may be very considerable, even when the partial pressure of the impurity is less than one-thousandth of a millimetre. This conclusion also applies to cases where the conductivity of a gas A is improved by the admixture of small quantities of a gas B. In these cases the ionizing potential of B is less than that of A, and the probability of an electron ionizing a molecule of B is very much greater than the probability of ionizing a molecule of A †.

We have recently made some further investigations on this effect, in order to estimate the amount of ionization which may be attributed to the effects of direct collisions of electrons with molecules of impurities in monatomic gases. The results will be published in another paper.

Various methods of purification of the argon were tried, and the method finally adopted was a modification of that devised by Soddy, where calcium was used to absorb the impurities ‡.

The gas, which was obtained from the British Oxygen Company, contained about 1 per cent. of nitrogen and traces of other impurities, and after being subjected to a preliminary purification by circulation over heated calcium, it was transferred at a pressure of about 400 mm. to a pyrex flask A of about 1 litre capacity, which had been heated to a high temperature for a considerable time before the argon was admitted.

It was found, in the course of this preliminary purification, that hydrogen was one of the most difficult gases

* J. S. Townsend, ' Motion of Electrons in Gases.' (Clarendon Press, Oxford.)

† This has already been pointed out in explaining the improvement of the conductivity of neon due to very small traces of impurity (J. S. Townsend & S. P. McCallum, Phil. Mag. v. p. 695 (1928)).

‡ F. Soddy, Proc. Roy. Soc. lxxvii. A, p. 429 (1906).

to eliminate.  The hydrogen was finally eliminated by passing the argon over palladium black, which had been heated to a high temperature for a long time.  A small quantity of palladium black was also placed in the storage flask A.

The apparatus finally adopted is shown in fig. 1.

Quantities of the gas from the flask A could be admitted to the pyrex tube B, which contained charcoal. A quartz tube (C) was used to contain small lumps of calcium.  It was connected to the tube B by a graded join, Q – P, of quartz to pyrex.   The tube C was connected through a pair of co-axial quartz tubes (D) to the quartz tube E, in which measurements of the electrical properties of the gas were made.  Thus the part of the apparatus from Q to E was of quartz only, without any taps.

Fig. 1.
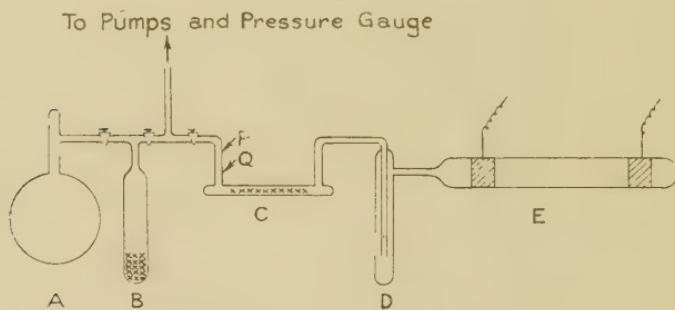


Diagram of apparatus.

The three pyrex taps shown in the figure were lubricated with apiezon grease, and the usual precautions were taken to prevent the diffusion of mercury vapour from the pumps and gauges.  The tubes D acted as a trap for any particles of calcium that might be blown from the tube C by accidental rushes of gas.

The argon in the tube E was circulated several times over the heated calcium by cooling the charcoal in B to the temperature of liquid air (thus ab orbing the argon in the charcoal), and then allowing the charcoal to attain room temperature.  The rate at which the argon was circulated over the calcium could be regulated by the rate of cooling of the charcoal in B.

It is of importance not to have any taps between the calcium tube and the tube E.   When this precaution was

observed the argon remained perfectly pure, and no change in its electrical properties were observed however long it was kept in the tube E.

Before any gas was admitted to the apparatus all the quartz parts and the calcium parts were heated to a bright red heat for long periods while a high vacuum was maintained. At the same time the pyrex parts and the charcoal were heated for some hours to about 300° C. When these precautions were observed it was found possible to obtain the argon so that no impurities could be detected in the spectrum between $\lambda = 7000$ Å and $\lambda = 2000$ Å. at pressures as high as 150 mm.

An idea of the efficacy of the calcium in absorbing impurities may be gained from the following experiment. Fresh calcium was placed in C, and the apparatus evacuated. When the pressure was about one-thousandth of a millimetre the pumps were turned off and the calcium was heated. Gases occluded by the calcium were thus driven off and the pressure rose to about 20 mm., but on further heating these gases were absorbed by the calcium, and within a short space of time the pressure fell again to its previous level.

The appearance of the luminous column of the high-frequency discharge in argon is quite different from that in helium or neon. In a tube 1·7 cm. in diameter containing argon at pressures less than 10 mm., the discharge in pure argon is striated with all currents and light purple in colour. At these pressures the striations consist of luminous ovals about 1 cm. in length, separated by comparatively dark spaces 0·5 to 1 cm. in length.

At higher pressures the discharge may assume two forms. With large currents the discharge is still striated, the size of the striations decreasing as the pressure is increased. With small currents the tube is filled with a uniform luminous column which is whitish in colour.

In a tube 3 cm. in diameter it was found difficult to obtain the current sufficiently intense to produce the striated form when the pressure exceeded about 80 mm. When the striations were obtained they were very small, consisting of a chain of bright points about 1 mm. in diameter stretching between the electrodes. At these high pressures the chain of striations did not lie along the axis of the tube as at low pressures, but were above the axis along an arc, as is shown in fig. 2.

The effect was much the same as the bowing of an arc in air at atmospheric pressure, and is due to the heating of the gas along the line of the discharge, which is considerable at these high pressures. When the tube was placed in a vertical position the effect was absent, and the striations lay in a straight line along the axis of the tube *.

Some authors † have stated that the existence of striations is due to impurities. In these experiments we found that when the argon was spectroscopically pure the luminous column was always striated when the pressure was small, but when the pressure was high the column was uniform with small currents, and striated with large currents. It was also found that at the lower pressures

Fig. 2.
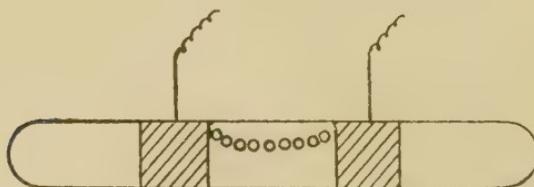


Striations in argon at 150 mm. pressure.

the striated column became uniform when a small quantity of nitrogen was introduced.

In these experiments the force was measured by the method used by Townsend and Donaldson ‡. External electrodes in the form of sleeves were used, and the potentials between the electrodes were measured for the smallest current it was possible to maintain with uniform luminosity in the tube. The force Z was then obtained from the differences between the potentials required to maintain a small current with the electrodes at different distances apart.

The electromotive force E between the electrodes, which were in the form of lead sleeves 4 cm. wide, consists of two parts : the fall of potential in the luminous column,

* A short description of these forms of the discharge has been published in ' Nature ' (McCallum, Klatzow, and Keyston, cxxx. p. 810 (1932)).

† See K. Darrow, ' Electrical Phenomena in Gases,' and Emeleus, ' Conduction of Electricity in Gases.' (Methuen.)

‡ J. S. Townsend and R. H. Donaldson, Phil. Mag. v. p. 178 (1928).

and the fall of potential near the electrodes. It has been shown by other experiments * that the total potential E may be expressed in terms of the force Z in the positive column, the distance $x$ between the electrodes, the wave-length $\lambda$ of the oscillations, and the current I in the tube, by the relation

$$E = Z(x+a) + b\lambda I,$$

where $a$ and $b$ are constants for any particular pressure, Z is independent of the wave-length of the oscillations and the intensity of the current in the tube, provided the latter is not sufficiently large to cause appreciable heating of the gas. When the current is small the term $b\lambda I$ is small compared with $Z(x+a)$, so that Z can be found very accurately.

Since the discharge is striated when the pressure is below about 5 mm., even with the smallest currents, and the electric force Z is different in the luminous and dark portions of the striated column, it is necessary to measure the average force in the tube by adjusting the distances between the electrodes so as to have different numbers of complete striations between them. At a pressure of 1 millimetre the potential between two striations was 8 volts in the tube 1·7 cm. in diameter, and 5·5 volts in the tube 3 cm. in diameter.

Measurements of the force Z along the axis of the tube in volts per cm. were made for wave-lengths from 40 metres to 220 metres in two tubes 1·7 and 3 cm. in diameter respectively. It was found that the force Z was very nearly independent of the wave-length in this range. The results of experiments with oscillations of 220 metres wave-length are given in Table I., where Z is the root-mean-square value of the force in volts per centimetre.

These results are shown in the curves of fig. 3, where the ordinates are proportional to the force Z and the abscissæ to the pressure $p$.

It was found that the current caused considerable heating of the gas when the pressure was high. This was much more marked in the wide tube than in the small tube. For this reason the values of the gauge readings of the pressures above 30 mm. indicate only approximately the values of the density of the gas in the

* J. S. Townsend and W. Nethercot, Phil. Mag. vii. p. 600 (1929).

path of the discharge. The true density of the gas in the discharge may be very much less than that indicated

TABLE I.

| Diameter of tube = 1·7 cm. | | | Diameter of tube = 3 cm. | | |
|---|---|---|---|---|---|
| $p$. | $Z$. | $Z/p$. | $p$. | $Z$. | $Z/p$. |
| 1 | 3 | 3 | 1·1 | 2·1 | 2 |
| 5·5 | 7·8 | 1·4 | 4·7 | 6·2 | 1·3 |
| 14 | 15·4 | 1·1 | 9·0 | 10·8 | 1·2 |
| 28 | 25 | 0·89 | 26·5 | 22·9 | 0·9 |
| 39 | 32 | 0·82 | 46 | 26 | 0·6 |
| 54 | 37 | 0·68 | 82 | 40 | 0·5 |
| 100 | 52 | 0·52 | 140 | 54 | 0·4 |

Fig. 3.
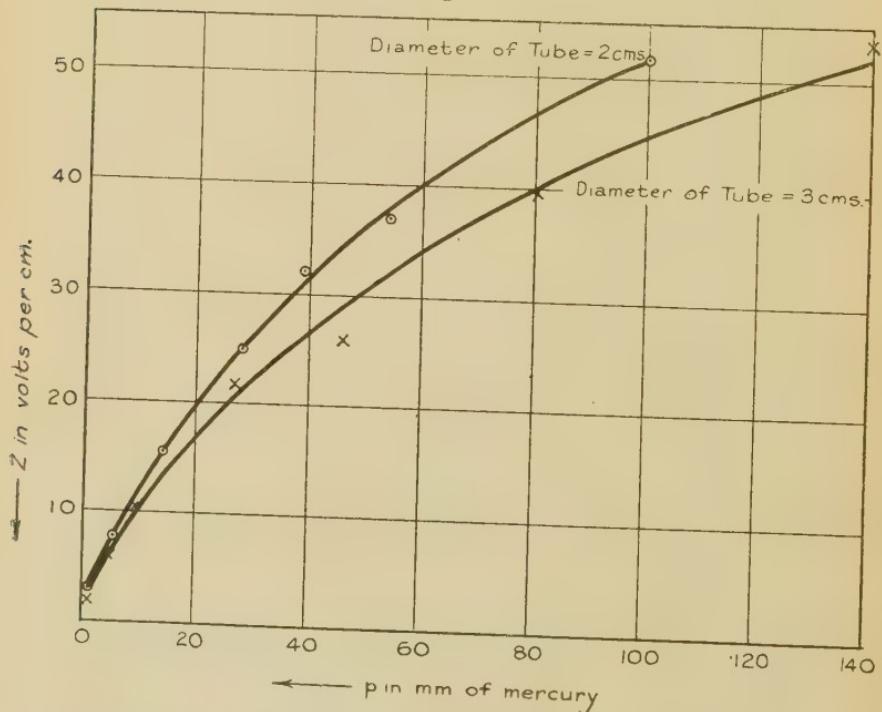


by the gauge when this is greater than 30 mm. The bending over of the curves in fig. 3 at the higher pressures may be attributed in large part to this effect. The heating of the gas and the consequent convection currents

also cause the luminosity to be unevenly distributed about the axis of the tube.

It is of interest to compare the forces in the uniform discharge in argon at different pressures with the forces in helium and neon. The forces Z corresponding to different pressures $p$ are given for the three gases by the curves of fig. 4, the diameter of the tubes being 3 cm.

It is seen that at any given pressure the value of $Z/p$ or argon is less than that for helium and greater than hat for neon. For example, at 20 mm. pressure the

Fig. 4.
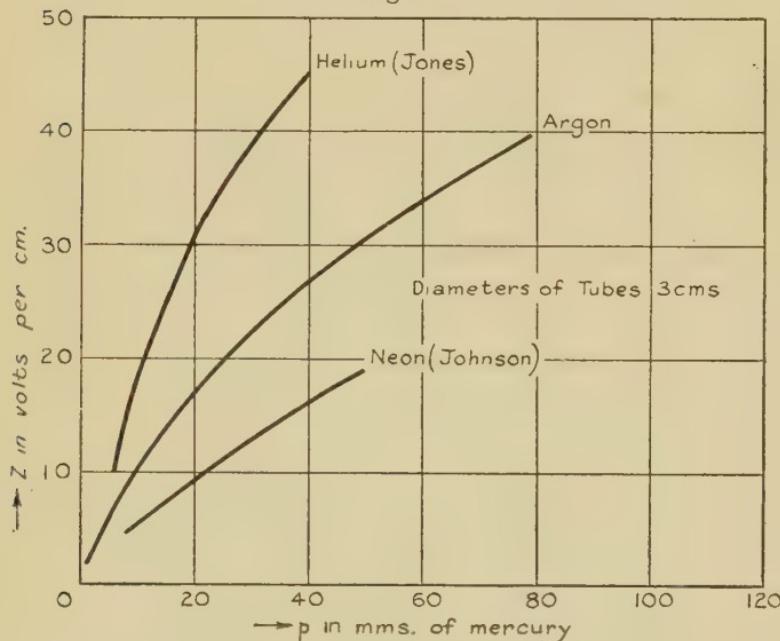
values of $Z/p$ for helium, argon, and neon are 1·6, 0·9, and 0·45 respectively. The mean energy of agitation of the electrons in the three gases for these values of $Z/p$ are 3·2, 10·0, and 4·6 volts respectively *, so that in the discharges at 20 mm. pressure the mean energy of agitation of the electrons in argon is much greater than in helium or in neon.

There is a considerable difference in the field of force in the discharges in the different gases due to the radial force required to control the diffusion of the electrons

* 'Motions of Electrons in Gases,' by J. S. Townsend. (Clarendon Press, Oxford. 1925.)

to the sides of the tube.   If R be the force along a radius at a distance $(r)$ from the axis of the tube, due to the charge in the gas, it has been shown * that the potential difference $\int_0^a R\,dr$ between  a point  on the  axis and a point at a distance $(a)$ from the  axis is $2\,\dfrac{E}{3}\,\log\left(\dfrac{A}{A_1}\right)$, where E is the mean energy of agitation of the electrons in volts, A the concentration of electrons at the axis, and $A_1$ the concentration at a point $r=a$.   If $a$ be the radius of the tube, and if it be assumed that the concentration at a point near the surface be half that at the axis, then the potential $\int_0^a R\,dr$ is 0·46 E.    Thus, at 20 mm. pressure the values of the mean radial force in argon, helium, and neon are 4·6, 1·5, and 2·1 volts. per cm. respectively.

The concentration of the luminosity of the discharge in argon along the axis of the tube is probably due, in a large measure, to the high value for the radial force, since the mean energy of agitation of the electrons must be considerably less at points near the surface of the tube than at points near the axis, and hence most of the ionization and excitation of the spectral lines will take place near the axis of the tube.

In helium and neon, as well as in argon, the energy of agitation of the electrons increases as the pressure of the gas diminishes, so that at pressures less than 1 millimetre there is a comparatively large force along the radius, and it has been observed that at such pressures the luminous column is striated in these gases.

The potentials necessary to start discharges in argon in tubes 2 cm. and 3 cm. in diameter were also measured. These potentials depend on the distance $x$ between the electrodes, the wave-length $\lambda$ of the oscillations, and the pressure $p$ of the gas.

The results obtained with the two tubes are given in Table II., where the distance $x$ is given in  cm., the wave-length $\lambda$ in metres, and the pressure $p$ in mm.; the values of the potentials given are the root-mean-square values.

For pressures up to about 3 mm. in the 2 cm. tube the potentials required to start the discharge increase with the wave-length ;  but at higher pressures the values

* J. S. Townsend, Phil. Mag. xiii. p. 750 (1932).

of the starting potentials are almost independent of the wave-length. At any pressure above that at which the starting potential is a minimum the difference $V_4 - V_2$ between $V_4$, the starting potential when the electrodes

## TABLE II.

| | Diameter of tube = 2 cm. | | | | | |
|---|---|---|---|---|---|---|
| $p$ in mm. | $\lambda = 80$ metres. | | $\lambda = 160$ metres. | | $\lambda = 220$ metres. | |
| | $x = 2$ cm. | $x = 4$ cm. | $x = 2$ cm. | $x = 4$ cm. | $x = 2$ cm. | $x = 4$ cm. |
| | volts. | volts. | volts. | volts. | volts. | volts. |
| 0·16 | 105 | 125 | 160 | 158 | 172 | 184 |
| 0·36 | 98 | 144 | 135 | 160 | 142 | 171 |
| 0·74 | 106 | 175 | 128 | 190 | 134 | 185 |
| 1·02 | 112 | 192 | 132 | 210 | 131 | 221 |
| 1·48 | 130 | 220 | 146 | 235 | 151 | 230 |
| 2·98 | 172 | 296 | 178 | 304 | 180 | 302 |
| 4·12 | 204 | 348 | 203 | 334 | 202 | 329 |
| 5·48 | 219 | 388 | 221 | 384 | 213 | 376 |
| 8·9 | 275 | 492 | 287 | 484 | 280 | 477 |
| 12·6 | — | — | 326 | 565 | 313 | 567 |

| | Diameter of tube = 3 cm.    $\lambda = 220$ metres. | |
|---|---|---|
| $p$ in mm. | $x = 2$ cm. | $c = 4$ cm. |
| 0·12 | 201 volts. | 167 volts. |
| 0·16 | 151 ,, | 141 ,, |
| 0·23 | 131 ,, | 131 ,, |
| 0·36 | 114 ,, | 124 ,, |
| 0·53 | 112 ,, | 136 ,, |
| 0·77 | 111 ,, | 143 ,, |
| 1·08 | 124 ,, | 160 ,, |
| 2·24 | 149 ,, | 226 ,, |
| 3·21 | 169 ,, | 262 ,, |
| 4·14 | 185 ,, | 282 ,, |
| 4·73 | 199 ,, | 303 ,, |

are 4 cm. apart, and $V_2$, that when they are 2 cm. apart, is independent of the wave-length.

The general properties of the spectrum of the discharges in argon at different pressures will be described in another paper.

We wish to thank Professor Townsend for much helpful advice.

LXVII. *On the Slow Rotation of an Anchor Ring in an Infinite Viscous Liquid.* By Prof. F. E. RELTON, D.Sc., M.A., *Royal School of Engineering, Giza, Egypt* *.

### *Summary.*

THE tore has slow steady rotation about its axis of figure in a viscous liquid. The path of each particle of the liquid is taken as a circle ; this assumption is shown to be equivalent to the ignoration of the inertia terms. The analysis is conducted in orthogonal curvilinear ring coordinates, and the solution of the problem depends on Laplace's equation for the vector velocity. Solutions are obtained in terms of associated Legendre functions of degree $\frac{1}{2}$ and integral order. The usual dyadics lead to an expression for the resisting couple. Numerical illustrations and an approximate formula are given. The paper is complementary to one that appeared in Phil. Mag. xi. p. 129 (Jan. 1931).

### *The Coordinates.*

1. The transition from ordinary cylindrical coordinates $r$, $z$, $\phi$ to ring coordinates $\lambda$, $\theta$, $\phi$ is made by the relation

$$exp(\lambda+i\theta)=(b+r+iz)/(b-r-iz), \quad . \quad . \quad . \quad (1)$$

where $b$ is a constant. From this we deduce

$$r=b \sinh \lambda/(\cosh \lambda+\cos \theta), \quad . \quad . \quad . \quad (2)$$

and for the arc-element we have

$$ds^2=dx^2+dy^2+dz^2$$
$$=dr^2+dz^2+(r \cdot d\phi)^2$$
$$=(rh)^2(d\lambda^2+d\theta^2)+(r \cdot d\phi)^2, \quad . \quad . \quad . \quad (3)$$

where $h=\operatorname{cosech} \lambda$ and is essentially positive in the first quadrant. One other convenient abbreviation that will be employed systematically is $\mu=\coth \lambda$, so that

$$1-\mu^2 = -h^2=d\mu/d\lambda. \quad . \quad . \quad . \quad . \quad (4)$$

The geometrical interpretation is that if A, B be two points on the $r$-axis in the meridian plane, their distances from the origin being $\pm b$ respectively, then for any other point P in the same meridian plane, $\exp \lambda = \text{PB/PA}$,

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* Communicated by the Author.

and the angle APB $=\pi-\theta$. Constant $\lambda$ gives a family of coaxal circles with A, B as limiting points ; constant $\theta$ gives the conjugate family. As P traverses the $z$-values $\infty$, 0, $-\infty$ the angle $\theta$ ranges through $\pi$, 0, $-\pi$. At an infinite distance and all down the $z$-axis, we have $\exp \lambda$ unity, whilst $\mu$ is infinite. At the point A we have $\exp \lambda$ infinite, whilst $\mu$ is unity. By revolving the meridian plane about the $z$-axis, the $\lambda$-circles generate anchor rings. Evidently $\lambda$, $\theta$, $\phi$ are orthogonal curvilinear coordinates.

### The Unit Vectors.

2. We take a right-handed system of unit vectors l, m, n in the directions of increasing $\lambda$, $\theta$, $\phi$ respectively. This means that m is moved to n by a rotation which appears to be counterclockwise when viewed from the end of l. Accordingly the direction of increasing $\phi$ is clockwise when viewed from the positive end of the $z$-axis. By the properties of unit vectors we have

$$\frac{\partial^2 \mathbf{n}}{\partial \phi^2} = -\mathbf{n}, \quad \ldots \quad \ldots \quad (5)$$

and from (3) the geometry of the configuration gives

$$\frac{\partial \mathbf{n}}{\partial \phi} = -\frac{1}{rh}\frac{\partial r}{\partial \lambda} - \frac{\mathbf{m}}{rh}\frac{\partial r}{\partial \theta}, \quad \ldots \quad \ldots \quad (6)$$

together with

$$\frac{\partial \mathbf{n}}{\partial \lambda} = 0 = \frac{\partial \mathbf{n}}{\partial \theta}. \quad \ldots \quad \ldots \quad (7)$$

The differential operator " del " is given by

$$\nabla = \frac{1}{rh}\left\{ \mathbf{l}\frac{\partial}{\partial \lambda} + \mathbf{m}\frac{\partial}{\partial \theta} + h\mathbf{n}\frac{\partial}{\partial \phi} \right\}, \quad \ldots \quad (8)$$

and Laplace's operator is

$$\nabla^2 = \frac{1}{r^2 h^2}\left\{ \frac{\partial}{\partial \lambda}\left( r\frac{\partial}{\partial \lambda} \right) + \frac{\partial}{\partial \theta}\left( r\frac{\partial}{\partial \theta} \right) + \frac{\partial}{\partial \phi}\left( rh^2\frac{\partial}{\partial \phi} \right) \right\}. \quad (9)$$

### The Steady Motion.

3. If body forces are absent, the equation

$$\nu\nabla^2 \mathbf{v} - \nabla p/\rho = \mathbf{v} \cdot \nabla \mathbf{v} \quad \ldots \quad \ldots \quad (10)$$

gives the steady motion of an incompressible liquid, of density $\rho$ and kinematical viscosity $\nu$, where $p$ is the

pressure and $\mathbf{v}$ is the vector velocity. For the steady motion we shall make the assumption which is usually made in the analogous case of the rotating sphere, that the particles of the fluid describe horizontal circles about the $z$-axis with uniform velocity. Thus we have $\mathbf{v}=\mathbf{n}v$ and $\partial v/\partial\phi=0$.

By (8), the right side of (10) gives

$$\mathbf{v} \cdot \nabla\mathbf{v}=\frac{v}{r}\frac{\partial}{\partial\phi}(\mathbf{n}v)=\frac{v^2}{r}\frac{\partial\mathbf{n}}{\partial\phi},$$

which, by (6), is expressible in terms of $\mathbf{l}$ and $\mathbf{m}$. The symmetry of the motion dictates $\partial p/\partial\phi=0$, so that $\nabla p$ is also expressible in terms of $\mathbf{l}$ and $\mathbf{m}$. Hence (10) requires

$$\nabla^2\mathbf{v}=0=\frac{\mathbf{n}}{r^3h^2}\left\{\frac{\partial}{\partial\lambda}\left(r\frac{\partial v}{\partial\lambda}\right)+\frac{\partial}{\partial\theta}\left(r\frac{\partial v}{\partial\theta}\right)-rh^2v\right\}. \quad (11)$$

The substitution $w=vr^{\frac12}$ converts this last equation to

$$\frac{\partial^2w}{\partial\lambda^2}+\frac{\partial^2w}{\partial\theta^2}-\tfrac{3}{4}h^2w=0. \quad . \quad . \quad . \quad (12)$$

## The Solution of the Equation.

4. We seek for solutions of (12) in Fourier form by putting $w=\mathrm{L}\cos k\theta$, where L is a function of $\lambda$ alone, to be determined from

$$\frac{d^2\mathrm{L}}{d\lambda^2}-(k^2+\tfrac{3}{4}h^2)\mathrm{L}=0. \quad . \quad . \quad . \quad (13)$$

This is really the associated Legendre equation : for since $d/d\lambda=(d\mu/d\lambda)\cdot(d/d\mu)$, we can utilize (4) to write (13) as

$$(1-\mu^2)\frac{d}{d\mu}\left\{(1-\mu^2)\frac{d\mathrm{L}}{d\mu}\right\}-\{k^2-\tfrac{3}{4}(1-\mu^2)\}\mathrm{L}=0, \quad (14)$$

showing that the degree is $\frac{1}{2}$ and the order $k$. For the present problem I shall employ the solutions $\mathrm{P}_{\frac12}^{-k}(\mu)$ and $\mathrm{Q}_{\frac12}^{k}(\mu)$, and in order to particularize, I purpose adopting Hobson's definitions

$$\left.\begin{aligned}
\mathrm{P}_n^m(\mu)&=\frac{1}{\Gamma(1-m)}\left(\frac{\mu+1}{\mu-1}\right)^{\frac12 m}\mathrm{F}(-n,n+1;\ 1-m;\ \tfrac{1}{2}-\tfrac{1}{2}\mu),\\
\mathrm{Q}_n^m(\mu)&=e^{m i\pi}\frac{\Gamma(n+m+1)\Gamma(\tfrac12)}{2^{n+1}\Gamma(n+1\tfrac12)}\frac{(\mu^2-1)^{\frac12 m}}{\mu^{n+m+1}}\mathrm{F}(\tfrac12 n+\tfrac12 m+\tfrac12,\\
&\qquad\qquad\qquad\qquad \tfrac12 n+\tfrac12 m+1;\ n+1\tfrac12;\ \mu^{-2}).
\end{aligned}\right\}.$$
$$. \quad . \quad . \quad (15)$$

The P-function is suitable in the neighbourhood of the limiting point and tends to zero when $\mu$ tends to unity. The Q-function is suitable in a region extending to infinity, where $\mu$ is infinite. In the present case, where the degree is $\frac{1}{2}$ and the order integral, the Q-function tends to zero at infinity, the magnitude being of the order $0^{1\frac{1}{2}}$. Suitable solutions of (11) are accordingly

$$vr^{\frac{1}{2}} = w = \Sigma_{k=0}^{\infty} A_k P_{\frac{1}{2}}^{-k}(\mu)^{\cos}_{\sin} k\theta, \quad . \quad . \quad . \quad (16)$$

where the $A_k$ are arbitrary constants, yet to be determined, and the $P^{-k}$ can be replaced by $Q^k$.

### The Liquid Core.

5. Assuming that the surface of the tore is defined by $\lambda = c$, let us suppose that it rotates about its axis of figure, the $z$-axis, with constant angular velocity $\omega$. The assumed absence of slip at the surface provides the boundary condition

$$vr^{\frac{1}{2}} = w = \omega r^{3/2}. \quad . \quad . \quad . \quad . \quad (17)$$

Let us denote by $\gamma$ the surface value of $\mu$, and assume in the first case that the viscous liquid is present as a core to the anchor ring. We then have to determine the arbitrary constants from the boundary condition

$$\omega \left( \frac{b \sinh c}{\cosh c + \cos \theta} \right)^{3/2} = \Sigma_{k=0}^{\infty} A_k P_{\frac{1}{2}}^{-k}(\gamma) \cos k\theta. . \quad (18)$$

and it is obvious, from the symmetry of the motion, that the sine terms are not required on the right. If, now, we put

$$(\cosh c + \cos \theta)^{-3/2} = \frac{1}{2}a_0 + a_1 \cos \theta + a_2 \cos 2\theta \ldots . \quad (19)$$

we have *

$$a_k = \frac{1}{\pi} \int_0^{2\pi} \frac{\cos k\theta . d\theta}{(\cosh c + \cos \theta)^{3/2}}$$

$$= (-)^k 4 \left( \frac{\operatorname{cosech}^3 c}{\pi} \right)^{\frac{1}{2}} \Gamma(k + 1\tfrac{1}{2}) P_{\frac{1}{4}}^{-k}(\gamma) . \quad . \quad (20)$$

It thus appears that the value of the constant $A_k$ is independent of the size of the ring. The physical interpretation is that, in the steady state, the viscous core rotates with the ring as if solid.

* Relton, Journ. Lond. Math. Soc., Jan. 1932.

### The Infinite Liquid.

6. When the ring is immersed in a liquid which extends to infinity, where it is at rest, the solution in terms of Q-functions is needed,

$$vr^{\frac{1}{2}}=w=\varSigma_{k=0}^{\infty}\mathrm{B}_k\mathrm{Q}_{\frac{1}{2}}^k(\mu)\cos k\theta. \quad . \quad . \quad . \quad (21)$$

The arbitrary constants $\mathrm{B}_k$ are to be determined as before from the boundary condition of no slip, giving

$$\omega\left(\frac{b\sinh c}{\cosh c+\cos\theta}\right)^{3/2}=\varSigma_{k=0}^{\infty}\mathrm{B}_k\mathrm{Q}_{\frac{1}{2}}^k(\gamma)\cos k\theta. \quad . \quad (22)$$

On expanding the left side as a Fourier series, this leads to

$$\mathrm{B}_k=(-)^k4\pi^{-\frac{1}{2}}\omega b^{3/2}\varGamma(k+1\tfrac{1}{2})\mathrm{P}_{\frac{1}{2}}^{-k}(\gamma)/\mathrm{Q}_{\frac{1}{2}}^k(\gamma). \quad . \quad (23)$$

Thus the arbitrary constants, and hence the velocity distribution, are completely determined, and it will be shown in the next section that an expression is derivable for the resisting couple. Nevertheless the pressure distribution is not determinable. The first term in (10) is an n-vector, whilst the other two are expressible in terms of l and m. On equating the respective components we have

$$\frac{1}{\rho}\frac{dp}{d\lambda}=\frac{v^2}{r}\frac{dr}{d\lambda}, \quad \frac{1}{\rho}\frac{dp}{d\theta}=\frac{v^2}{r}\frac{dr}{d\theta},$$

whence, after a little reduction, on eliminating $p$

$$\frac{\partial(v,\,r)}{\partial(\lambda,\,\theta)}=0.$$

On the present hypothesis, $v$ is not in general a function of $r$ alone, and the equations for determining $p$ are inconsistent. The explanation of the anomaly, such as it is, was given by Stokes in the case of the rotating sphere ; the body acts as a centrifuge, and a circulatory motion in the meridian plane should be superimposed on the assumed motion. The usual way out of the impasse is to assume that the velocity is small enough to permit the ignoration of the inertia terms, which leaves $p$ constant.

### The Resistance.

7. For the stress-dyadic $\Psi$ we have in the usual notation

$$\Psi=\varSigma\mathrm{T}_{11}\mathrm{ll}+\varSigma\mathrm{T}_{12}(\mathrm{lm}+\mathrm{ml}),$$

and for $\Phi$, the self-conjugate part of the rate-of-strain dyadic, we have

$$\Phi = \tfrac{1}{2}(\nabla\mathbf{v}+\mathbf{v}\nabla),$$

where

$$\nabla\mathbf{v}=\left(\frac{\mathbf{l}}{rh}\frac{\partial}{\partial\lambda}+\frac{\mathbf{m}}{rh}\frac{\partial}{\partial\theta}+\frac{\mathbf{n}}{r}\frac{\partial}{\partial\phi}\right)\mathbf{n}v$$

$$=\mathbf{ln}\frac{1}{rh}\frac{\partial v}{\partial\lambda}+\mathbf{mn}\frac{1}{rh}\frac{\partial v}{\partial\theta}-\mathbf{nl}\frac{v}{r^2h}\frac{\partial r}{\partial\lambda}-\mathbf{nm}\frac{v}{r^2h}\frac{\partial r}{\partial\theta}$$

from (6). Hence

$$2\Phi=(\mathbf{ln}+\mathbf{nl})\frac{1}{h}\frac{\partial}{\partial\lambda}\left(\frac{v}{r}\right)+(\mathbf{mn}+\mathbf{nm})\frac{1}{h}\frac{\partial}{\partial\theta}\left(\frac{v}{r}\right).$$

In the absence of dilatation, the linear connexion between the dyadics is

$$\Psi=-p\mathbf{I}+2\sigma\Phi,$$

where $\mathbf{I}$ is the idem-factor and $\sigma$ is the ordinary viscosity. The stress across a plane normal to $\mathbf{l}$ is

$$\mathbf{l}.\Psi=\mathbf{l}\mathrm{T}_{11}+\mathbf{m}\mathrm{T}_{12}+\mathbf{n}\mathrm{T}_{13},$$

so that the tangential resistance arises from

$$\mathrm{T}_{13}=\frac{\sigma}{h}\frac{\partial}{\partial\lambda}\left(\frac{v}{r}\right)=\frac{\sigma}{h}\frac{\partial}{\partial\lambda}(wr^{-3/2}).$$

The resisting couple is thus seen to be

$$\int_0^{2\pi}\mathrm{T}_{13}2\pi r\,.\,r\,.\,rh\,d\theta$$

taken over the surface of the tore, or

$$\text{Couple}=2\pi\sigma\int_0^{2\pi}r^3\frac{\partial}{\partial\lambda}(wr^{-3,2})d\theta$$

$$=2\pi\sigma\int_0^{2\pi}\left(r^{3/2}\frac{\partial w}{\partial\lambda}-w\frac{\partial}{\partial\lambda}r^{3/2}\right)d\theta. \qquad . \quad (24)$$

In order to evaluate this integral we proceed as follows. Each of the two terms enclosed by the bracket under the integral sign is the product of two Fourier series. We can therefore utilize Parseval's theorem

$$\int_0^{2\pi}(\tfrac{1}{2}f_0+f_1\cos\theta+f_2\cos 2\theta+...)(\tfrac{1}{2}g_0+g_1\cos\theta$$

$$+g_2\cos 2\theta+...)d\theta=\pi(\tfrac{1}{2}f_0g_0+f_1g_1+f_2g_2+...).$$

Using (21) and (23) for $w$, and noting from (20) that

$$r^{3/2} = \Sigma_{k=0}^{\infty} b_k P_{\frac{1}{2}}^{-k}(\mu) \cos k\theta,$$

where $\qquad b_k = (-)^k 4\pi^{-\frac{1}{2}} b^{3/2} \Gamma(k+1\frac{1}{2}),$

the couple in (24) can be written

$$2\pi^2 \sigma \Sigma_{k=0}^{\infty} B_k b_k \left\{ P_{\frac{1}{2}}^{-k}(\gamma) \frac{d}{dc} Q_{\frac{1}{2}}^{k}(\gamma) - Q_{\frac{1}{2}}^{k}(\gamma) \frac{d}{dc} P_{\frac{1}{2}}^{-k}(\gamma) \right\}. \quad (25)$$

The value of $\{ \ \}$ in (25) is $(-)^k$. On inserting the values of $B_k$, $b_k$, the expression for the couple becomes

$$32\pi b^3 \sigma \omega \Sigma_{k=0}^{\infty} (-)^k \{ \Gamma(k+1\frac{1}{2}) \}^2 P_{\frac{1}{2}}^{-k}(\gamma) / Q_{\frac{1}{2}}^{k}(\gamma). \quad (26)$$

### Numerical Results.

8. In order to express the size of the ring in terms of the adopted symbols, suppose the generating circle $\lambda = c$ crosses the $r$-axis at distances $b+a$, $b-\beta$ from the origin. We then have by definition

$$(2b+a)/a = \exp c = (2b-\beta)/\beta,$$

whence the ratio

$$R = \frac{\text{external radius}}{\text{internal radius}} = \frac{b+a}{b-\beta} = \coth^2 \tfrac{1}{2}c.$$

From this it follows that

$$\gamma = (R+1)/2R^{\frac{1}{2}}, \quad R = \{ \gamma + (\gamma^2-1)^{\frac{1}{2}} \}^2.$$

Now $\gamma = \coth c > 1$, but the excess of $\gamma$ over unity is quite small over a fairly wide range of rings ; the excess, in fact, does not become marked until the ring is relatively thick. Thus if $R = 1\cdot 1$ we have $\gamma = 1\cdot 001$, and if $R = 1\cdot 134$, $\gamma = 1\cdot 002$. On bringing the excess to the next order of quantities, we find that $\gamma = 1\cdot 01$ gives $R = 1\cdot 33$, whilst $\gamma = 1\cdot 02$ gives $R = 1\cdot 5$, by which time the thickness of the ring is half the internal radius. The value $R = 3$, when the thickness of the ring equals the diameter of the aperture, requires $\gamma = 1\cdot 155$. The value $\gamma = 3$, which is critical in the sense that it delimits the convergence of the P-function as defined, corresponds to $R = 33\cdot 97$, and the ring is almost closed, the diameter of the aperture being then a mere 3 per cent. of the external diameter of the ring.

Adverting now to the general term under the summation sign in (26), it is a matter of mere algebraical reduction to show that it can be written

$$\left(\frac{8\gamma^3}{\pi}\right)^{\frac{1}{2}}\left(\frac{\gamma}{\gamma+1}\right)^k\frac{\Gamma(k+1\frac{1}{2})}{\Gamma(k+1)}\frac{\mathrm{F_P}}{\mathrm{F_Q}}, \quad \cdot \quad \cdot \quad \cdot \quad (27)$$

where

$$\mathrm{F_P}=\mathrm{F}(-\tfrac{1}{2}, 1\tfrac{1}{2} \; ; \; k+1 \; ; \; \tfrac{1}{2}-\tfrac{1}{2}\gamma),$$

$$\mathrm{F_Q}=\mathrm{F}(\tfrac{1}{2}k+\tfrac{3}{4}, \tfrac{1}{2}k+1\tfrac{1}{4} \; ; \; 2 \; ; \; \gamma^{-2}).$$

For values of $\gamma$ slightly in excess of unity, the series for $\mathrm{F_P}$ is rapidly convergent and differs but little from unity. Unfortunately, on the other hand, $\mathrm{F_Q}$ in the same circumstances is very slowly convergent, and the more so as $k$ increases. It appears, therefore, that over a wide range of rings we can made a good approximation by replacing $\mathrm{F_P}$ by unity for all values of $k$, at the same time replacing $\mathrm{F_Q}$ by its asymptotic value

$$\frac{\Gamma(2)\Gamma(k)}{\Gamma(\tfrac{1}{2}k+\tfrac{3}{4})\Gamma(\tfrac{1}{2}k+1\tfrac{1}{4})}(1-\gamma^{-2})^{-k}.$$

This in turn, by the use of Legendre's duplication formula, can be expressed as

$$\frac{2^{k+\frac{1}{2}}\Gamma(k)}{\pi^{\frac{1}{2}}\Gamma(k+1\tfrac{1}{2})}\left(\frac{\gamma^2}{\gamma^2-1}\right)^k,$$

whence it follows that the general term (27) is asymptotic to

$$\frac{\gamma^{3/2}}{2^{k-1}}\left(\frac{\gamma-1}{\gamma}\right)^k\frac{\{\Gamma(k+1\tfrac{1}{2})\}^2}{\Gamma(k)\Gamma(k+1)}$$

for values of $\gamma$ close to unity. The part of this that depends upon $k$ can be examined, for large values of $k$, by means of Stirling's formula

$$\Gamma(k)\rightarrow k^k e^{-k}(2\pi/k)^{\frac{1}{2}}.$$

The application of this shows that

$$\{\Gamma(k+1\tfrac{1}{2})\}^2/2^{k-1}\Gamma(k)\Gamma(k+1)\rightarrow(k+1\tfrac{1}{2})^2/2^{k-1},$$

which tends to zero with increasing $k$. The terms in (26) are therefore ultimately more convergent than the terms $\{(\gamma-1)/\gamma\}^k$, and it appears that a fair approximation can be made by limiting ourselves to the first term.

This first term, however, requires separate consideration. A comparison of (20) and (21) gives

$$B_0 Q_{\frac{1}{2}}^0(\gamma) = 2\omega b^{3,2} \pi^{-\frac{1}{2}} \Gamma(1\tfrac{1}{2}) P_{\frac{1}{2}}^0(\gamma) = \omega b^{3,2} P_{\frac{1}{2}}^0(\gamma),$$

whilst in (25) we have

$$b_0 = 4b^{3/2} \pi^{-\frac{1}{2}} \Gamma(1\tfrac{1}{2}) = 2b^{3/2}.$$

Hence the first term in (26) is

$$4\pi^2 b^3 \sigma \omega P_{\frac{1}{2}}^0(\gamma)/Q_{\frac{1}{2}}^0(\gamma).$$

On making $\gamma \to 1$ we have $P_{\frac{1}{2}}^0(\gamma) \to 1$, whilst

$$Q_{\frac{1}{2}}^0(\gamma) = \frac{\Gamma(1\tfrac{1}{2})\Gamma(\tfrac{1}{2})}{2^{1\frac{1}{2}}\Gamma(2)} \gamma^{-3/2} F(\tfrac{3}{4}, 1\tfrac{1}{4}; 2; \gamma^{-2}) \to \tfrac{1}{2}\gamma^{-3/2} \log\left(\frac{\gamma^2}{\gamma^2 - 1}\right),$$

so that the first term in (26) is asymptotic to

$$\frac{8\pi^2 b^3 \sigma \omega \gamma^{3/2}}{\log\ \{\gamma^2/(\gamma^2 - 1)\}}.$$

If we ignore $\gamma^{3/2}$ as differing but little from unity, and $\log \gamma^2$ as differing but little from zero, we can write the approximate formula for the resisting couple as

$$\frac{8\pi^2 b^3 \sigma \omega}{-\log (2f)},$$

where $f$, the excess of $\gamma$ over unity, is small. It is easily seen that the formula is dimensionally correct, and that the couple tends to zero with $f$, as it should.

For thicker rings, where the excess of $\gamma$ over unity is too large to permit the application of approximate methods, there is nothing for it but to calculate the functions. No tables exist, and the work must be continued until the ignored terms are cumulatively negligible. By dint of calculating over two hundred and fifty terms, each to eight significant figures, I find that with $\gamma = 1\tfrac{1}{2}$ the resisting couple is

$$32\pi b^3 \sigma \omega(\cdot 5861 + \cdot 6765 + \cdot 2907 + \cdot 1052 + \cdot 0320 + \cdot 0097$$
$$+ \cdot 0032 + \cdot 0010 \ldots) = 32\pi b^3 \sigma \omega \times 1 \cdot 704.$$

With $\gamma = 2$ the corresponding figures are

$$32\pi b^3 \sigma \omega(2 \cdot 329 + 3 \cdot 425 + 2 \cdot 111 + 1 \cdot 164 + \cdot 590 + \cdot 281$$
$$+ \ \cdot 127 + \cdot 045 \ldots) = 32\pi b^3 \sigma \omega \times 10 \cdot 123.$$

The domination of the leading term noticeably declines.

LXVIII. *Some Problems concerned with Inverted Prolate Spheroids.*

*To the Editors of the Philosophical Magazine.*

GENTLEMEN,—

I BEG to be allowed to make a brief remark concerning the interesting research of Dr. D. M. Wrinch on " Some Problems concerned with Inverted Prolate Spheroids," published in the Philosophical Magazine, No. 95 (December 1932).

In the expression of the *associated function* $\psi$, on p. 1065, the derivatives of the spherical functions may be eliminated by means of the formula

$$\frac{2n+1}{n(n+1)}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu} = \mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu) \quad . \quad . \quad (1)$$

This formula may be deduced as a particular case of a more general one valid for metaspherical functions (see Niels Nielsen, ' Théorie de fonctions métasphériques,' Paris, 1911, p. 55, formula 4).

Consider, for example, the general expression of the potential function $\phi$ by means of the *spherical harmonics*

$$\phi = \sum_n \left(\mathrm{C}_n r^n + \mathrm{D}_n \frac{1}{r^{n+1}}\right)\mathrm{P}_n(\mu),$$

in which $\mathrm{C}_n$ and $\mathrm{D}_n$ are arbitrary constants. The corresponding associated function is

$$\psi = \sum_n \left(\frac{\mathrm{C}_n}{n+1}r^{n+1} - \frac{\mathrm{D}_n}{n}\frac{1}{r^n}\right)(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu},$$

and, therefore, on account of (1),

$$\psi = \sum_n \frac{1}{2n+1}\left(\mathrm{C}_n n r^{n+1} - \mathrm{D}_n\frac{n+1}{r^n}\right)(\mathrm{P}_{n-1}-\mathrm{P}_{n+1}).$$

Doing the same for the *prolate spheroid* considered by Dr. Wrinch, we find

$$\phi = \sum_n \mathrm{C}_n\mathrm{P}_n(\mu)\mathrm{P}_n(\nu),$$

$$\psi = -\sum_n \frac{\mathrm{C}_n}{n(n+1)}(1-\mu^2)\frac{d\mathrm{P}_n}{d\mu}\cdot(\nu^2-1)\frac{d\mathrm{P}_n}{d\nu}$$

$$= \sum_n \mathrm{C}_n\frac{n(n+1)}{(2n+1)^2}[\mathrm{P}_{n-1}(\mu)-\mathrm{P}_{n+1}(\mu)][\mathrm{P}_{n-1}(\nu)-\mathrm{P}_{n+1}(\nu)].$$

(In all these formulæ $P_n$ may be substituted by $Q_n$, spherical function of the second kind.)

Expressions similar to those derived above may be obtained as well for harmonics of oblate spheroid and for *conical functions.* For the latter the formula becomes

$$\phi = \frac{1}{\sqrt{r}} \sum_k \left[ C_k \cos(k \log r) + D_k \sin(k \log r) \right] K_k(\mu)$$

(see Bateman, 'Partial Differential Equations,' Cambridge and New York, 1932, p. 382), and therefore, by aid of (1), substituting $-\frac{1}{2} + ik$ in place of $n$ and omitting the factor $i/4$,

$$\psi = \sqrt{r} \sum_k \left[ \left( \frac{C_k}{k} - 2D_k \right) \cos(k \log r) \right.$$
$$\left. + \left( \frac{D_k}{k} + 2C_k \right) \sin(k \log r) \right] \left[ K_{k-1}(\mu) - K_{k+1}(\mu) \right].$$

The above expressions simplify the numerical calculation of the associated functions.

<div style="text-align:center">Yours faithfully,</div>

Turin,                    I. OPATOWSKI, Dr. Ing., Dr. Mat.
      16th January. 1933.

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<div style="text-align:center">LXIX. <i>Notices respecting New Books.</i></div>

*Introduction to Theoretical Physics.* By MAX PLANCK. Translated by HENRY L. BROSE. In 5 Volumes. (Macmillan.) I. *General Mechanics* [pp. ix+272 (1933). Price 12s.]. II. *Mechanics of Deformable Bodies* [pp. vii+234 (1932). Price 10s. 6d.]. III. *Theory of Electricity and Magnetism* [pp. xii+247 (1932). Price 10s. 6d.]. IV. *Theory of Light* [pp. vii+216 (1932). Price 10s. 6d.]. V. *Theory of Heat* [pp. viii+301 (1932). Price 12s.].

THIS treatise on theoretical physics is based on lectures given by Planck to students at Berlin. The translation is from the latest German editions, the first editions of the successive volumes having appeared at intervals from 1916 to 1932. The field covered is that of classical physics, and the unique value of the work lies in the manner in which the parts of this field are systematically presented in their relatedness to the whole. Considering the scope, the bulk is not large : for the number of volumes is offset by the fact that the individual volumes, with some three hundred

words to a page of text, are of a length which compares favourably with other books on the separate branches, dealing with a similar range of subject matter. Although references are sometimes made in later to earlier volumes of the series, each volume is complete in itself. For the present purpose, however, it is convenient to consider them in their relation to each other, for there is a unifying outlook underlying them all, and the order corresponds to the development of a systematic scheme.

In the first volume, the aim, as stated in the preface, is to present mechanics as it has evolved step by step, the theorems being derived by methods which follow the physical ideas most closely, and which approximate to those which were used originally, or could have been used. Passing from the simple to the more complex, the mechanics of the material point is first considered, and then the statics and dynamics of systems of material points constituting rigid bodies. The arguments are illustrated by well chosen examples, and most of the standard problems receive adequate discussion. There is an excellent presentation of the Lagrangian and Hamiltonian equations, and of the problems of relative motion. Constraints are considered in the last chapter, which concludes with a general discussion of the top.

Proceeding to the next stage of complexity, in the second volume deformable bodies are considered. The sections deal with the general laws of motion, with very small deformations (vibrations and waves), and with finite deformations (irrotational and vortex motion, without and with friction). A full treatment is given of the effect of the viscosity of a medium on the motion of a sphere in it. The first two books give a singularly lucid and concise account of the fundamentals of dynamics, which is sufficient to enable the student to proceed with confidence to more specialized treatises.

A new range of phenomena is introduced in the third volume. The link between mechanics and electromagnetism is obtained by placing the principle of conservation of energy in the forefront. The treatment adopted, as the only possible one consistent with uniformity and completeness, is predominantly deductive. Although the starting-point is not in individual empirical facts, many examples are given, and a clear distinction is made between matters of definition and of experience. The treatment given is that based on the principle of contiguous action. A rather unusual procedure is followed in basing the derivation of the fundamental equations on the energy principle and Poynting's law of energy flow. Statical and stationary phenomena are then considered, while the third section is concerned with quasi-

stationary and dynamical processes, culminating in an
excellent discussion of the difficulties in connexion with
moving bodies which led to the principle of relativity.

The treatment of light is mainly restricted to those parts
of the subject which can be dealt with by the classical theory
as applied to continuous bodies. In the first section on
isotropic bodies, reflexion, refraction, interference, polariza-
tion, geometrical optics, and diffraction are considered.
In the second, a good account of crystal optics is given.
In the last section the atomic standpoint is introduced, and
the classical dipole oscillator theory of dispersion is clearly
presented. Finally, the relation between geometrical optics
and classical mechanics is discussed, and the manner in which
wave mechanics may be approached is outlined.

In the final volume on heat, the principle of conservation
of energy has to be supplemented by the principle of the
second law of thermodynamics, which involves the special
peculiarity of thermal processes in connexion with direction
of change. The sections deal with the thermodynamics,
conduction of heat, heat radiation, and atomic theory and
the theory of quanta. The discussion of the question of the
assignment of statistical weights leads to an admirable account
of the origin of the quantum theory.

This summary will give a general idea of the content
of these volumes. Although much of the treatment is
mathematical, the actual knowledge of mathematics assumed
is not beyond that which the average student of physics
may be reasonably expected to have acquired. The presen-
tation throughout has remarkable ease and lucidity, which
has been well maintained in the admirable straightforward
translation. The volumes give an account of a vast range
of physics as it appears to one who has played a prominent
part in its development, and who has spared no pains to
present it as a logical and satisfying whole. There can be
no doubt as to the success in the achievement of what has been
attempted.

It is, perhaps, desirable to consider what particular value
this work may have for the student and teacher of physics.
Can it largely replace other text-books ? Will it serve
as a work of reference ? As a work of reference no claims
would be made, for its value largely depends on its being
read, as it were, in blocks rather than in fragments. By its
very nature it is one sided—it is theoretical physics that
is considered,—and to many the divorce of theory and
experiment, necessary though it may be for special purposes,
must seem unfortunate in what professes to be an elementary
treatise for students. The method of presentation is largely
that which has become customary, with a few exceptions,

in text-books on mechanics. The historical development of physics is subordinated to the logical development of an argument. Now although much of theoretical physics may have reached a state in which it may be presented as a logical argument, the historical development has not been one in which argument has been elaborated to be checked by experiment merely at a few convenient points. When it is presented in that way, it is liable to give an erroneous impression of the character of science. From this point of view the work is, perhaps, hardly suitable as a basis for a student who is attempting to gain an appreciation of physics as it has developed and as it is still developing. But to one who has an adequate background, these volumes will be invaluable in enabling him to benefit from the coordinating grasp of the subject as a whole which is revealed, and in enabling him to appreciate more fully the power and scope of the methods which are involved.

Student and teacher alike will derive both enjoyment and stimulus from a cursory reading of the whole as well as from a detailed study of particular parts. The energy of the translator and the enterprise of the publishers in making this survey of physics by an acknowledged leader more widely accessible are deserving of high praise.     E. C. S.

*Statistical Methods for Research Workers* (4th edition, re-vised and enlarged). (No. V.—Biological Monographs and Manuals. Editors : F. A. E. CREW and D. WARD CUTLER.) By R. A. FISHER, Sc.D., F.R.S. [Pp. xiii+307.] (Edin-burgh : Oliver and Boyd. Price 15*s*. net.)

THE appeal of Dr. Fisher's well-known monograph, which is now in its fourth edition, is chiefly to research workers in the biological and allied sciences, and particularly to those whose data are drawn from samples too small to warrant the use of methods suitable for infinitely large samples. The author deals entirely with the application of statistical methods to actual problems, by means of examples worked out in detail to illustrate the points discussed. Proofs of the mathematical theorems on which the methods are based are not given ; reference must be made to the original papers if these are required. The disadvantage of this method of presentation lies in the stress laid on the definite problem chosen as an illustration ; to one unfamiliar with the technical nomenclature of this particular field of enquiry the points at issue may tend to be obscured, and the reader's judgment as to whether the process exemplified will apply to his own material be made more difficult. In most cases, however, the problems are quite straightforward, and probably the majority of those for whom the work is intended would find

no difficulty at all on this score, in which case the inclusion of a large amount of mathematics would, from their point of view, be unnecessary and undesirable.

In this edition greater attention is paid to the idea of covariance, and in an additional section the analysis of covariance is discussed at some length. Apart from this, the chief alterations are in the use of the *k*-statistics in preference to the moments in the chapter on distributions, a change which simplifies the algebra involved when fitting an observed distribution to a theoretical frequency curve ; and in the substitution of an historical survey in place of an account of tables for testing significance. Seven important tables are given on folded sheets bound at the end of the book, as well as in the appropriate places in the text, thus allowing them to be taken out and mounted separately if desired. A list of sources used for data and methods and a bibliography containing the author's own statistical publications are also included at the end of the text.

*Recent Advances in Atomic Physics.* By GAETANO CASTEL-FRANCHI. Translated by W. S. STILES and J. W. T. WALSH. In two volumes. Vol. I *Atoms, Molecules, and Electrons* [pp. 300]. Vol. II. *Quantum Theory* [pp. 400]. (J. & A. Churchill. Price 15*s*. each volume.)

THERE is, apparently, a loud and persistent call from students in the upper forms of schools, as well as from University students, for informative and up-to-date works on physics. Such a book, for example, as ' Modern Physics,' by Professor H. A. Wilson, has been found exceedingly useful by many students as an introduction and guide to the higher reaches of the subject. This translation of Professor Castelfranchi's two volume text-book is another work of this type.

In two moderate sized books Professor Castelfranchi has covered almost the whole range of modern physics. It is not, therefore, to be expected that within so small a compass should be found more than the briefest outline and in some cases the barest mention, of topics which in some quarters might be regarded as deserving broader and deeper treatment.

It is, on the other hand, rather surprising that the author should have thought it worth while to devote so many valuable pages to considerations of a distinctly elementary character. In Vol. I., for example, the greater part of a chapter is given to simple physical optics and in Vol. II. several pages to practical television.

The books should appeal in particular to University students and to senior school students, as they give a broad and general view of the results of modern physics, stressing principles rather than the details of mathematical treatment.

The first book contains nine chapters, commencing with one of an historical character dealing with the history of the atom and molecule from the earliest times. Then follows a chapter on light, while the kinetic theory of gases receives treatment in Chapter III. In this chapter the simple derivatives of Boyle's Law and the meaning of van der Waals's expression are clearly elucidated and the distribution law of Maxwell enunciated. The usual elementary treatments of mean free path and molecular diameters are also adequately presented, the chapter ending with a brief reference to vacuum pumps and manometers.

Chapter IV., on thermodynamics, mentions probability relations, fluctuations, and the theory of emulsions.

Chapter V. gives a general account of the electron and its main properties in the light of kinetic theory, conductivity, thermionic emission. Electron diffraction is omitted. A concise account of isotopes ends this long chapter.

Chapters follow on X-rays and crystals, naturally only in outline—withal a most useful one. Seventy pages of radio-activity and nuclear physics complete the book.

The second volume is devoted, in the main, to those phenomena related by quantum theory.

Thermal radiation, spectra (including an account of the Stark and Zeeman effects), and the spinning electron receive attention in the first hundred pages. In the next hundred, specific heats, photoelectricity, and protons are duly treated.

Magnetism receives a chapter to itself, while the last hundred pages are devoted to wave mechanics and the new statistics. In this last section are included accounts of electron diffraction, indeterminacy, and the well-known applications of the new theories.

There are few errors of any import, and in general the treatment is very clear and concise, and praise is due not only to the author but also to the translators (who, by the way, are active workers at the National Physical Laboratory) for the skill with which their part of the work has been carried through.

The printing is first rate, the diagrams adequate, and the bibliographies at the end of the individual chapters excellent.

*The Mechanism of Creative Evolution.* By C. C. Hurst. (Cambridge University Press, 1932. Price 21s.)

This book gives a simple and elementary account of the position to-day in the domain of Genetics. It is nowadays widely claimed by geneticists and others that the gene is the unit of life and that " the discoveries of the chromosomes and genes and their interrelations prove to be as important to the biologist as the discoveries of the atoms and electrons and their interrelations are to the physicist " (to quote from

the introduction of the work at present under review). And indeed it seems undeniable that the recent advances in biology have reduced the distance of biology from the senior sciences very considerably indeed. There is, in fact, a sense in which genetics has crossed the threshold of molecular physics.

Genetics has evidently a tremendous importance, in that it has built a bridge whereby the discoveries of physics, possibly also the technique of physics, may become a prime factor in the progress of biological knowledge. We may even indulge in a fantasy of the future, when biology has become a partner with physics and the molecular problems of chromosome behaviour form part, and possibly the most complicated part, of molecular theory.

This book is recommended to physicists who care to read a modern account of the position of Genetics written on simple lines and supplemented by a wealth of beautiful pictures.

A PAPER entitled *Gravity Anomalies and the Structure of the Earth's Crust*, by Major E. A. GLENNIE, D.S.O., R.E., has reached us, as published by order of Brigadier R. H. Thomas, D.S.O., Surveyor General of India.

Its content is interesting as containing a considerable number of observations which, according to the writer, do not support "Isostasy" as generally accepted. But :— "Although isostasy as a fact is denied, as an illusion it persists ; hence the presentation of gravity results in the form of Hayford anomalies still remains the best method for universal application as a first step towards the investigations of the structure of the Earth's crust."     J. J.

## LXX. *Intelligence and Miscellaneous Articles.*

### UNION INTERNATIONALE DE CHIMIE.

THE Federal Council for Chemistry has received a communication from the Union Internationale de Chimie sending a copy of a report of the Committee on Thermochemistry and stating that it is proposed to publish an International Table of Thermochemical Data. The Union asks for publicity to be given to the request of the Secretary of the Committee that Authors who have published papers on thermochemistry within the last five years should send him two copies of their papers. Such copies should be forwarded to Dr. L. J. P. Keffler, The University, Liverpool.

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S. E. CARR,
*Assistant Secretary,*
*Federal Council for Chemistry.*

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*[The Editors do not hold themselves responsible for the views expressed by their correspondents.]*

Fig. 3.